Optical solitons and other solutions to the
Radhakrishnan-Kundu-Lakshmanan equation

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ABSTRACT

This paper studies the perturbed Radhakrishnan–Kundu–Lakshmanan (RKL) equation and its
special form, the generalized RKL equation. Kerr law nonlinearity is considered for both equa-
tions. The Riccati-Bernoulli sub-ODE technique is implemented to these nonlinear equations so
that their optical solitons and complex wave solutions are retrieved. In order to derive a sequence
of solutions of considered equations, Bäcklund transformation can be carried out. 3D and 2D
graphics of some solutions are depicted for chosen suitable parameters.

1. Introduction

Soliton theory has wide application in nonlinear physics fields with the inclusion of fluid dynamics, nonlinear fiber optics, plasma
physics, and optics; thus, it has attracted much attention from researchers for the past few decades [1–3]. There are many mathe-
matical models for propagation of soliton in the literature such as nonlinear Schrödinger’s equation [4,5], Chen–Lee–Liu equation [6],
Schrödinger–Hirota equation [7], Kundu–Eckhaus equation [8,9], Sasa–Satsuma equation [10–12], Manakov model [13,14], Kun-
du–Mukherjee–Naskar equation [15].

The gist of this paper is to examine the following perturbed and generalized RKL equations, given as.

In this paper, the following perturbed and generalized RKL equations are considered, given as

\[ iu_t + au_{xx} + \beta F(|u|^2)u = i\sigma u_x + i\mu\left\{ F(|u|^2)u \right\}_x + i\gamma u_{xxx}, \]

(1.1)

and

\[ iu_t + au_{xx} + \beta F(|u|^2)u = i\sigma F(|u|^2)u_x - i\gamma u_{xxx}. \]

(1.2)

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where $u(x, t)$ specifies the wave profile with independent variables $x$ and $t$ that indicate the spatial and temporal variables, respectively. $F$ is the function of nonlinearity type, which will be determined. $\alpha$ is defined as the group velocity dispersion term coefficient and $\beta$ is described as the nonlinearity coefficient. For the right-hand side of Eq. (1.1), $\sigma$ represents the dispersion of inter-modal, $i$ represents the self-steepening for short pulses coefficient, $\mu$ is defined as the coefficient of higher-order dispersion, and $\gamma$ is the third-order dispersion term. Moreover, the special case where $\sigma = 0$ and $\mu = 0$ of the perturbed RKL equation is represented in Eq. (1.2). However, two equations are distinctly modeled. To produce new optical solitons of the RKL equation with Kerr nonlinearity, a new generalized exponential rational function method (GERFM) is utilized in [16]. The same method is also used for acquiring kink-type solitons, traveling waves, and novel solitons with a complex form of the nonlinear RKL equation in [17]. In [18], with the help of the extended trial function method, singular, dark, and bright soliton solutions for the RKL model are retrieved. In [19], implementing the solitary wave ansatz technique, an exact 1-soliton solution of the generalized RKL equation is acquired. In order to derive dark, bright, singular, and dark-singular combo dispersive optical solitons for the RKL equation with full nonlinearity, sine-Gordon and $G'/G^2$-expansion schemes are applied in [20]. Bright, dark, periodic, and singular soliton solutions of the RKL equation by power-law nonlinearity are produced with the aid of the exp$(-\varphi(q))$ and the modified simple equation techniques in [21]. In [22], trial equation and modified simple equation schemes are implemented to extract optical soliton solutions for the perturbed RKL equation with considering Kerr law and power-law nonlinearities.

This study aims to study the optical soliton and complex wave solutions of the perturbed RKL equation and its special model that is generalized RKL equation with Kerr law nonlinearity utilizing the Riccati-Bernoulli sub-ODE method. Riccati-Bernoulli sub-ODE scheme is a successfully implemented technique to derive the exact solutions of the nonlinear partial differential equations (NLPDEs). As a literature review for this work, some of the substantial previous studies are examined. For example, the nonlinear fractional Klein-Gordon equation, the Eckhaus equation, the generalized Zakharov-Kuznetsov-Burgers equation, and the generalized Ostrovsky equation are constructed in [23]. Utilizing the Riccati-Bernoulli sub-ODE technique, singular and dark soliton solutions of nonlinear Schrödinger’s equation (NLSE) with spatio-temporal dispersion are retrieved in [24]. In [25], soliton solutions of the perturbed NLSE with quadratic-cubic nonlinearity are revealed with the help of Kudryashov’s technique and the Riccati-Bernoulli sub-ODE scheme.

The remaining parts of this paper are constructed as follows: In Section 2, we represent the definition of the Riccati-Bernoulli sub-ODE scheme. In Section 3, the applications of this technique for the perturbed RKL equation and generalized RKL equation with Kerr law nonlinearity are presented in order to obtain soliton and complex wave solutions of these equations. Furthermore, using the MAPLE software, 3D and 2D graphs of some acquired solutions are illustrated for chosen parameters in this section. In the last section, the conclusion is given.

2. Riccati Bernoulli sub-ODE method

In this section, we clarify the Riccati Bernoulli sub-ODE method. The Riccati Bernoulli sub-ODE method is a successfully implemented technique to derive the exact solutions of the nonlinear partial differential equations (NLPDEs). Any NLPDE, consisting of two independent variables $x$ and $t$, is expressed in the following form:

$$P(\theta, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \ldots) = 0. \quad (2.1)$$

**Step 1:** To acquire the solitary wave solutions of Eqs. (1.1)–(1.2), we utilize the following traveling wave transformation

$$\theta(x, t) = \theta(\zeta), \quad \zeta = k(x + vt), \quad (2.2)$$

in which $\theta(x, t) = \theta(\zeta)$ is an unknown function, $k$ represents the width of the traveling wave, and $v$ represents the velocity of the soliton. Then, Eq. (2.1) is turned into the following ODE:

$$P(\theta, \theta', \theta'', \ldots) = 0, \quad (2.3)$$

in which $\theta' = \frac{d\theta}{d\zeta}$, $\theta'' = \frac{d^2\theta}{d\zeta^2}$, and so on.

**Step 2:** Suppose that Eq. (2.3) is the solution of the Riccati-Bernoulli equation of the form:

$$\theta' = a_1 \theta + a_2 \theta^{2-n} + a_3 \theta^n, \quad (2.4)$$

in which $a_1$, $a_2$, $a_3$ and $m$ are constants. From Eq. (2.4), we can proceed the following derivatives,

$$\theta' = \theta^{-1+2m}(a_3 \theta^2 + a_1 \theta^{2m} + a_3 \theta^{1+n}) (-a_2(-2 + m)\theta^2 + a_1 m \theta^{2m} + a_1 \theta^{1+n}), \quad (2.5)$$

and

$$\theta'' = \theta^{-2+1+m}(a_1 \theta + a_2 \theta^{2-n} + a_3 \theta^n)(a_2(-2+m)(-3+2m)\theta^4$$

$$+a_2 (m(-1+2m)\theta^{4m} + a_1 a_3 (-3+m)(-2+m)\theta^{3+n} + (a_1^2 + 2a_2 a_3) \theta^{1+2m} + a_1 a_3 (1+m)\theta^{1+3m}) \quad (2.6)$$

The solutions of Eq. (2.4) are expressed as follows:

**Set 1:** When getting $m = 1$, Eq. (2.4) has the following solution
\[ \theta(\zeta) = K e^{\left( a_1 + a_2 + a_3 \right) \zeta}. \]  
(2.7)

**Set 2:** When getting \( m \neq 1 \), \( a_1 = 0 \), and \( a_3 = 0 \), Eq. (2.4) has the following solution
\[ \theta(\zeta) = (a_2(m - 1)(\zeta + K))^{\frac{1}{m}}. \]  
(2.8)

**Set 3:** When getting \( m \neq 1 \), \( a_1 = 0 \), and \( a_3 = 0 \), Eq. (2.4) has the following solution
\[ \theta(\zeta) = \left( K e^{(n-1)\zeta} - \frac{a_2}{a_1} \right)^{\frac{1}{n}}. \]  
(2.9)

**Set 4:** When getting \( m \neq 1 \), \( a_2 \neq 0 \), and \( a_1^2 - 4a_2a_3 < 0 \), Eq. (2.4) has the following solutions
\[ \theta(\zeta) = \left( -\frac{a_1}{2a_2} + \frac{\sqrt{4a_2a_3 - a_1^2}}{2a_2} \tan \left[ \frac{(1-m)\sqrt{4a_2a_3 - a_1^2}}{2}(\zeta + K) \right] \right)^{\frac{1}{m}}. \]  
(2.10)

and
\[ \theta(\zeta) = \left( -\frac{a_1}{2a_2} - \frac{\sqrt{4a_2a_3 - a_1^2}}{2a_2} \cot \left[ \frac{(1-m)\sqrt{4a_2a_3 - a_1^2}}{2}(\zeta + K) \right] \right)^{\frac{1}{m}}. \]  
(2.11)

**Set 5:** When getting \( m \neq 1 \), \( a_2 \neq 0 \), and \( a_1^2 - 4a_2a_3 > 0 \), Eq. (2.4) has the following solutions
\[ \theta(\zeta) = \left( -\frac{a_1}{2a_2} - \frac{\sqrt{a_1^2 - 4a_2a_3}}{2a_2} \tanh \left[ \frac{(1-m)\sqrt{a_1^2 - 4a_2a_3}}{2}(\zeta + K) \right] \right)^{\frac{1}{m}}. \]  
(2.12)

and
\[ \theta(\zeta) = \left( -\frac{a_1}{2a_2} - \frac{\sqrt{a_1^2 - 4a_2a_3}}{2a_2} \coth \left[ \frac{(1-m)\sqrt{a_1^2 - 4a_2a_3}}{2}(\zeta + K) \right] \right)^{\frac{1}{m}}. \]  
(2.13)

**Set 6:** When getting \( m \neq 1 \), \( a_2 \neq 0 \), and \( a_1^2 - 4a_2a_3 = 0 \), Eq. (2.4) has the following solution
\[ \theta(\zeta) = \left( \frac{1}{(2a_2(m-1)(\zeta + K) - a_3) \sqrt{K}} \right)^{\frac{1}{m}}. \]  
(2.14)

in which \( K \) is a constant.

**Step 3:** In the last step, if \( \theta \) and its derivatives are substituted into Eq. (2.3), we get a set of algebraic equations consisting of the power of \( \theta \). Putting the coefficients of each power of \( \theta \) to zero gives a system of algebraic equations for \( a_1, a_2, a_3, k, v \). If the parameters are substituted into Eqs. (2.7)–(2.14) are produced.

### 2.1. Bäcklund transformation

In this subsection, we will describe a Bäcklund transformation of the Riccati Bernoulli.

If \( \theta_n(\zeta) \) and \( \theta_{n-1}(\zeta) \) are solutions of Eq. (2.4), then we procure
\[ \frac{d\theta_n(\zeta)}{d\zeta} = \frac{d\theta_n(\zeta)}{d\theta_{n-1}(\zeta)} \frac{d\theta_{n-1}(\zeta)}{d\zeta} \frac{d\theta_{n-1}(\zeta)}{d\theta_{n-1}(\zeta)} (a_1\theta + a_2\theta^{2-n} + a_3\theta^n). \]  
(2.15)

From Eq. (2.15), we can write the following equality:
\[ \frac{d\theta_n(\zeta)}{a_1\theta + a_2\theta^{2-n} + a_3\theta^n} = \frac{d\theta_{n-1}(\zeta)}{a_1\theta_{n-1} + a_2\theta_{n-1}^{2-n} + a_3\theta_{n-1}^n}. \]  
(2.16)

When Eq. (2.16) is integrated with respect to \( \zeta \), we procure
\[ \theta_n(\zeta) = \left( \frac{-a_1C_1 + a_3C_2(\theta_{n-1}(\zeta))^{1-n}}{(a_1C_1 + a_2C_2 + a_3C_1(\theta_{n-1}(\zeta))^{1-n})} \right)^{\frac{1}{n}}, \]  
(2.17)

in which \( C_1 \) and \( C_2 \) are arbitrary constants. Eq. (2.17) is named as Bäcklund transformation. Utilizing Bäcklund transformation, we can derive infinite sequences of solutions for Eq. (2.1).
3. Applications

3.1. Applications of Riccati-Bernoulli sub-ODE method to Eq. (1.1)

The nonlinearity function $F$ is described as $F(y) = y$ for Kerr law nonlinearity. Thus, the model represented in Eq. (1.1) becomes

$$iu_t + au_{xx} + \beta|u|^2u = i\sigma u + i\kappa (|u|^2)_x + i\rho (|u|^2)_x u - i\mu x.$$  \hfill (3.1)

In order to get started, the following complex wave transformation is substituted into Eq. (3.1),

$$u(x,t) = U(\xi)e^{i\Omega(x,t)}, \quad \xi = x - vt, \quad \Omega(x,t) = -\kappa x + \omega t + \theta$$  \hfill (3.2)

then we have the following relations:

$$-(\sigma + v + 2\alpha x + 3\gamma^2)U - \frac{3\lambda + 2\mu}{3}U^3 + \gamma U' t = 0$$  \hfill (3.3)

from the imaginary part and

$$-(\omega + \alpha x^2 + \gamma x^3 + \sigma x)U + (\beta - \lambda x)U^3 + (\alpha + 3\gamma)U' t = 0$$  \hfill (3.4)

from the real part.

Since the function $U(\xi)$ satisfies both Eqs. (3.3) and (3.4), we can procure the following constraint conditions,

$$\frac{\omega + \alpha x^2 + \gamma x^3 + \sigma x}{\sigma + v + 2\alpha x + 3\gamma^2} = \frac{3(\beta - \lambda x)}{3\lambda + 2\mu} = \frac{\alpha + 3\gamma}{\gamma}$$  \hfill (3.5)

whenever

$$v = -\frac{8\gamma^3 + 8\kappa^2\gamma + 2\sigma\kappa^2 + 2\sigma^2 + \sigma^2 - \gamma\omega}{\alpha + 3\gamma} \quad \beta = -\frac{6\gamma\lambda + 6\kappa\mu + 3\alpha\lambda + 2\alpha\gamma}{3\gamma}$$  \hfill (3.6)

Eq. (3.4) will be utilized to get soliton solutions using the parameter Eq. (3.6). If $U$ and its derivatives are substituted into Eq. (3.4), and we set $m = 0$, we get the following equation,

$$\left(-\frac{6\gamma\lambda + 6\kappa\mu + 3\alpha\lambda + 2\alpha\gamma}{3\gamma} - \lambda x + 2(3\gamma + \alpha)\alpha_1^2\right)u^3 + 3\left(3\gamma + \alpha\right)\alpha_1\alpha_2u^2$$

$$+ (\omega - \alpha x^2 - \gamma x^3 - \sigma x + (3\gamma + \alpha)(\alpha_1^2 + 2\alpha_2\alpha_3))u + (3\gamma + \alpha)\alpha_1\alpha_3 = 0$$  \hfill (3.7)

If we collect all the coefficients of $U^i (i = 0, 1, 2, 3)$ and equate each to zero in Eq. (3.3), then we have the following system:

$U^0$ coefficient:

$$(3\gamma + \alpha)\alpha_1\alpha_3 = 0$$  \hfill (3.8)

$U^1$ coefficient:

$$(3\gamma + \alpha)\alpha_1\alpha_2 = 0$$  \hfill (3.9)

$U^2$ coefficient:

$$3(3\gamma + \alpha)\alpha_1\alpha_2 = 0$$  \hfill (3.10)

$U^3$ coefficient:

$$\left(-\frac{6\gamma\lambda + 6\kappa\mu + 3\alpha\lambda + 2\alpha\gamma}{3\gamma} - \lambda x + 2(3\gamma + \alpha)\alpha_1^2\right) = 0.$$  \hfill (3.11)

When solving the system of algebraic equations in Eqs. (3.8)–(3.11), the following cases are obtained. 

Set-1:

$$a_1 = 0, \quad a_2 = \frac{\sqrt{6}\sqrt{\gamma(3\lambda + 2\mu)}}{6\gamma}, \quad \omega = -\alpha x^2 - \gamma x^3 - \sigma x - \kappa a_1\sqrt{6}\sqrt{\gamma(3\lambda + 2\mu)} - \frac{\alpha a_1\sqrt{6}\sqrt{\gamma(3\lambda + 2\mu)}}{3\gamma}$$  \hfill (3.12)

If the parameters in Eq. (3.12) are substituted into (2.10)–(2.13), we acquire the following exact traveling wave solutions of Eq. (1.1):
\[u_{1,1}(x,t) = -\frac{1}{6} A_{1} \gamma \sqrt{6}
\]
\[
\tan \left( A_{1} \left( x + \frac{\left( 8\kappa \gamma^2 + 8\kappa^2 \sigma + 2\kappa \sigma \gamma + \sigma \alpha - \left( -\kappa^2 \alpha - \gamma \kappa^3 - \sigma \kappa - \kappa \alpha \right) 6 \sqrt{\gamma(3\lambda + 2\mu)} \right)}{3\kappa + \alpha} \right) + K \right)
\]
\[
e^\left( -e^{i \left( -\kappa^2 \alpha - \gamma \kappa^3 - \sigma \kappa - \kappa \alpha \right) 6 \sqrt{\gamma(3\lambda + 2\mu)} \right) \eta + \theta )
\]
\[
\sqrt{\gamma(3\lambda + 2\mu)}
\]

\[u_{1,2}(x,t) = \frac{1}{6} A_{1} \gamma \sqrt{6}
\]
\[
\cot \left( A_{1} \left( x + \frac{\left( 8\kappa \gamma^2 + 8\kappa^2 \sigma + 2\kappa \sigma \gamma + \sigma \alpha - \left( -\kappa^2 \alpha - \gamma \kappa^3 - \sigma \kappa - \kappa \alpha \right) 6 \sqrt{\gamma(3\lambda + 2\mu)} \right)}{3\kappa + \alpha} \right) + K \right)
\]
\[
e^\left( -e^{i \left( -\kappa^2 \alpha - \gamma \kappa^3 - \sigma \kappa - \kappa \alpha \right) 6 \sqrt{\gamma(3\lambda + 2\mu)} \right) \eta + \theta )
\]
\[
\sqrt{\gamma(3\lambda + 2\mu)}
\]

in which \( A_{1} = \sqrt{\frac{6a_{3} \gamma \sqrt{\gamma(3\lambda + 2\mu)}}{\gamma(3\lambda + 2\mu)}} \cdot \sqrt{\gamma(3\lambda + 2\mu)} > 0 \) and \( a_{3} \gamma < 0 \) for the presence of solutions.

Fig. 1. 3D and 2D graphs of \(|u_{1,2}(x,t)|^2\) for Eq. (1.1) for the values \( a_{3} = -1, \kappa = -1.5, \sigma = -0.5, \lambda = 0.5, \gamma = -0.5, \theta = 1 \) and \( K = 1. \)
Fig. 2. 3D and 2D graphs of $|u_{1,3}(x,t)|^2$ for Eq. (1.1) for the values $a_3 = 1, \kappa = 0.5, \alpha = 2, \sigma = 0.5, \lambda = 0.5, \gamma = -0.5, \theta = 0.3$ and $K = 1.$
Set-3:

\[
\begin{align*}
 u_{2,1}(x,t) &= \frac{-\sqrt{2}}{2} \frac{\sqrt{\omega + \alpha^2 + \gamma^3 + \sigma \kappa}}{(3\gamma + \alpha)} \tan \left( \frac{\sqrt{2} \sqrt{\omega + \alpha^2 + \gamma^3 + \sigma \kappa}}{3\gamma + \alpha} \left( x + \frac{(8\gamma^3 + 8\gamma^2 \alpha^2 + 2\alpha^2 + 2\kappa \gamma + \alpha^2 - \gamma \omega)t}{3\gamma + \alpha} + K \right) \right), \\
 u_{2,2}(x,t) &= \frac{1}{2} \frac{\sqrt{\omega + \alpha^2 + \gamma^3 + \sigma \kappa}}{(3\gamma + \alpha)} \sqrt{6} \cot \left( \frac{\sqrt{2} \sqrt{\omega + \alpha^2 + \gamma^3 + \sigma \kappa}}{3\gamma + \alpha} \left( x + \frac{(8\gamma^3 + 8\gamma^2 \alpha^2 + 2\alpha^2 + 2\kappa \gamma + \alpha^2 - \gamma \omega)t}{3\gamma + \alpha} + K \right) \right), \\
 u_{2,3}(x,t) &= \frac{1}{2} \frac{\sqrt{2} \sqrt{\omega + \alpha^2 + \gamma^3 + \sigma \kappa}}{(3\gamma + \alpha)} \sqrt{6} \tanh \left( \frac{\sqrt{2} \sqrt{\omega + \alpha^2 + \gamma^3 + \sigma \kappa}}{3\gamma + \alpha} \left( x + \frac{(8\gamma^3 + 8\gamma^2 \alpha^2 + 2\alpha^2 + 2\kappa \gamma + \alpha^2 - \gamma \omega)t}{3\gamma + \alpha} + K \right) \right), \\
 u_{2,4}(x,t) &= \frac{1}{2} \frac{\sqrt{2} \sqrt{\omega + \alpha^2 + \gamma^3 + \sigma \kappa}}{(3\gamma + \alpha)} \sqrt{6} \coth \left( \frac{\sqrt{2} \sqrt{\omega + \alpha^2 + \gamma^3 + \sigma \kappa}}{3\gamma + \alpha} \left( x + \frac{(8\gamma^3 + 8\gamma^2 \alpha^2 + 2\alpha^2 + 2\kappa \gamma + \alpha^2 - \gamma \omega)t}{3\gamma + \alpha} + K \right) \right),
\end{align*}
\]

with \((\omega + \alpha^2 + \gamma^3 + \sigma \kappa)(3\gamma + \alpha) > 0\) for the presence of solutions.

\[
\begin{align*}
 u_{2,1}(x,t) &= \frac{-\sqrt{2}}{2} \frac{\sqrt{\omega + \alpha^2 + \gamma^3 + \sigma \kappa}}{(3\gamma + \alpha)} \tan \left( \frac{\sqrt{2} \sqrt{\omega + \alpha^2 + \gamma^3 + \sigma \kappa}}{3\gamma + \alpha} \left( x + \frac{(8\gamma^3 + 8\gamma^2 \alpha^2 + 2\alpha^2 + 2\kappa \gamma + \alpha^2 - \gamma \omega)t}{3\gamma + \alpha} + K \right) \right), \\
 u_{2,2}(x,t) &= \frac{1}{2} \frac{\sqrt{2} \sqrt{\omega + \alpha^2 + \gamma^3 + \sigma \kappa}}{(3\gamma + \alpha)} \sqrt{6} \cot \left( \frac{\sqrt{2} \sqrt{\omega + \alpha^2 + \gamma^3 + \sigma \kappa}}{3\gamma + \alpha} \left( x + \frac{(8\gamma^3 + 8\gamma^2 \alpha^2 + 2\alpha^2 + 2\kappa \gamma + \alpha^2 - \gamma \omega)t}{3\gamma + \alpha} + K \right) \right), \\
 u_{2,3}(x,t) &= \frac{1}{2} \frac{\sqrt{2} \sqrt{\omega + \alpha^2 + \gamma^3 + \sigma \kappa}}{(3\gamma + \alpha)} \sqrt{6} \tanh \left( \frac{\sqrt{2} \sqrt{\omega + \alpha^2 + \gamma^3 + \sigma \kappa}}{3\gamma + \alpha} \left( x + \frac{(8\gamma^3 + 8\gamma^2 \alpha^2 + 2\alpha^2 + 2\kappa \gamma + \alpha^2 - \gamma \omega)t}{3\gamma + \alpha} + K \right) \right), \\
 u_{2,4}(x,t) &= \frac{1}{2} \frac{\sqrt{2} \sqrt{\omega + \alpha^2 + \gamma^3 + \sigma \kappa}}{(3\gamma + \alpha)} \sqrt{6} \coth \left( \frac{\sqrt{2} \sqrt{\omega + \alpha^2 + \gamma^3 + \sigma \kappa}}{3\gamma + \alpha} \left( x + \frac{(8\gamma^3 + 8\gamma^2 \alpha^2 + 2\alpha^2 + 2\kappa \gamma + \alpha^2 - \gamma \omega)t}{3\gamma + \alpha} + K \right) \right),
\end{align*}
\]

with \((\omega + \alpha^2 + \gamma^3 + \sigma \kappa)(3\gamma + \alpha) < 0\) for the presence of solutions. (Figs. 3 and 4).

Set-3:

\[
a_1 = 0, \quad a_2 = \frac{\sqrt{6} \sqrt{3\gamma + 2\mu}}{6\gamma}, \quad a_3 = 0, \quad \omega = -\alpha^2 - \gamma^3 - \sigma \kappa
\]

If the parameters in Eq. (3.14) are substituted into (2.10)–(2.13), we acquire the following exact traveling wave solution of Eq. (1.1):

\[
u_3,1(x,t) = \frac{-\sqrt{6} e^{(( - \kappa x + ( - \alpha^2 - \gamma^3 - \sigma \kappa)t + \theta)I)} \left( x + \frac{(8\gamma^3 + 8\gamma^2 \alpha^2 + 2\alpha^2 + 2\kappa \gamma + \alpha^2 - \gamma \omega)t}{3\gamma + \alpha} + K \right)}{\left( x + \frac{(8\gamma^3 + 8\gamma^2 \alpha^2 + 2\alpha^2 + 2\kappa \gamma + \alpha^2 - \gamma \omega)t}{3\gamma + \alpha} + K \right)}.
\]

3.2. Applications of Riccati-Bernoulli Sub-Ode Method to Eq. (1.2)

The nonlinearity function \(F\) is described as \(F(y) = y\) for Kerr law nonlinearity. Thus, the model represented in Eq. (1.2) reduces to

\[
iu + au_{xx} + \beta |u|^2 u = i\lambda \left( |u|^2 \right)_{x} - i\mu u_{xxx}.
\]

In order to get started, the following complex wave transformation is substituted into Eq. (2),

\[
u(x,t) = U(\xi)^{i\Omega(x,t)}, \quad \xi = x - vt, \quad \Omega(x,t) = \kappa x + \omega t + \theta
\]

then we have the following relations:
from the imaginary part and

\[(\omega + \alpha \kappa^2 + \gamma \kappa^3) U - (\beta - \lambda \kappa) U^3 - (\alpha + 3\gamma \kappa) U'/t = 0,\] (3.17)

from the real part.

As the function $U(\xi)$ satisfies both Eqs. (3.16) and (3.17), we obtain the following constraint conditions

\[\frac{\omega + \alpha \kappa^2 + \gamma \kappa^3}{\nu + 2\alpha \kappa + 3\gamma \kappa^2} = \frac{\beta - \lambda \kappa}{\lambda} = -\frac{\alpha + 3\gamma \kappa}{\gamma},\]

Fig. 4. 3D and 2D graphs of $|u_{3,3}(x,t)|^2$ for Eq. (1.1) for the values $\kappa = -1, \alpha = 0.5, \sigma = 5, \lambda = 0.5, \mu = 2.5, \gamma = -1, \theta = 0.3$ and $K = 0.25$. 
whenever
\[ v = \frac{\gamma \omega + \gamma \alpha^2 + \gamma^2 \kappa^5}{\alpha + 3\gamma \kappa} - 2\alpha \kappa - 3\gamma \kappa^2, \quad \beta = -\frac{\lambda \alpha - 2\gamma \lambda \kappa}{\gamma}. \] (3.18)

Eq. (3.16) will be utilized to acquire soliton solutions using the parameter Eq. (3.18).

Set-1:
\[ a_2 = \frac{\lambda}{2\gamma} a_1 = 0, \quad \omega = \frac{2\lambda}{\gamma} a_1 \alpha + 3\sqrt{2\gamma} a_1 \kappa - \gamma \kappa^3 - \alpha^2 \] (3.19)

Using these parameters in Eqs. (2.10)–(2.13), we obtained the following solutions that satisfy the Eq. (1.2):

\[ u_{1,1}(x,t) = \sqrt{2\gamma \lambda a_1} \tan \left( \frac{\sqrt{2}}{2} \sqrt{2\gamma \lambda a_1} \right) \left( x - \left( \frac{3\sqrt{2} a_1 (\gamma \kappa + \alpha^3) \sqrt{\gamma}}{3 \gamma \kappa + \alpha} - 2\alpha \kappa - 3\gamma \kappa^2 \right) t + K \right) \]
\[ \times e^{-\left( -\alpha \gamma, \frac{\sqrt{2} \gamma \lambda a_3}{\sqrt{2} \gamma \lambda a_3} \omega^2 \right) \gamma} \],

\[ u_{1,2}(x,t) = -\sqrt{2\gamma \lambda a_1} \cot \left( \frac{\sqrt{2}}{2} \sqrt{2\gamma \lambda a_1} \right) \left( x - \left( \frac{3\sqrt{2} a_1 (\gamma \kappa + \alpha^3) \sqrt{\gamma}}{3 \gamma \kappa + \alpha} - 2\alpha \kappa - 3\gamma \kappa^2 \right) t + K \right) \]
\[ \times e^{-\left( -\alpha \gamma, \frac{\sqrt{2} \gamma \lambda a_3}{\sqrt{2} \gamma \lambda a_3} \omega^2 \right) \gamma} \],

with \( \lambda \gamma > 0 \) and \( a_3 \lambda > 0 \) for the presence of solutions.

Fig. 5. 3D representation of square of modulus (a) for solution \( u_{1,2}(x,t) \) with \( a = -1, \gamma = -0.3, \kappa = 3, a_3 = -0.5, \lambda = -0.5, \theta = -1, K = 1 \) and (b) for solution \( u_{1,3}(x,t) \) with \( a = -1, \gamma = 0.5, \kappa = 1, a_3 = -1, \lambda = 0.5, \theta = 0.3, K = 1 \).
\[ u_{1,1}(x,t) = -\sqrt{\frac{\sqrt{2}y\alpha_1}{\gamma^2}} \tanh \left( \frac{\sqrt{2}y\alpha_1}{\gamma} \right) \left( x - \left( \frac{3\sqrt{2}\alpha_3(\gamma x^3 + \alpha^2 + \omega)}{3\gamma x + \alpha} \right) t + K \right) \]
\[ \times e^{\left( i\gamma x^3 - \frac{\pi}{2} \right)} , \]
\[ u_{1,4}(x,t) = -\sqrt{\frac{\sqrt{2}y\alpha_1}{\gamma^2}} \coth \left( \frac{\sqrt{2}y\alpha_1}{\gamma} \right) \left( x - \left( \frac{3\sqrt{2}\alpha_3(\gamma x^3 + \alpha^2 + \omega)}{3\gamma x + \alpha} \right) t + K \right) \]
\[ \times e^{\left( i\gamma x^3 - \frac{\pi}{2} \right)} , \]

with \( \lambda > 0 \) and \( a_3 \lambda < 0 \) for the presence of solutions. (Fig. 5).

**Set-2:**

\[ a_2 = \frac{\sqrt{2}y\lambda}{2\gamma}, \quad a_1 = 0, \quad a_3 = \frac{\sqrt{2}y(\gamma x^3 + \alpha^2 + \omega)}{\sqrt{\gamma(x + \alpha)}}. \]  

(3.20)

Substituting these parameters into Eqs. (2.10)–(2.13), we obtained the following solutions that satisfy the Eq. (1.2):

\[ u_{2,1}(x,t) = \frac{\sqrt{\gamma x^3 + \alpha^2 + \omega}}{\sqrt{\gamma(x + \alpha)}} \sqrt{2} \tanh \left( \frac{\sqrt{\gamma x^3 + \alpha^2 + \omega}}{6\gamma x + 2\alpha} \right) \left( x - \left( \frac{\gamma^2 x^3 + \gamma x^2 + \gamma^2\omega}{3\gamma x + \alpha} \right) t + K \right) \]
\[ \times e^{\left( i\omega x^3 - \frac{\pi}{2} \right)} , \]
\[ u_{2,2}(x,t) = -\frac{\sqrt{\gamma x^3 + \alpha^2 + \omega}}{\sqrt{\gamma(x + \alpha)}} \sqrt{2} \cot \left( \frac{\sqrt{\gamma x^3 + \alpha^2 + \omega}}{6\gamma x + 2\alpha} \right) \left( x - \left( \frac{\gamma^2 x^3 + \gamma x^2 + \gamma^2\omega}{3\gamma x + \alpha} \right) t + K \right) \]
\[ \times e^{\left( i\omega x^3 - \frac{\pi}{2} \right)} . \]

Fig. 6. (a) 3D representation of \( |u_{2,1}|^2 \) and (b) 2D graphics of \( |u_{2,2}|^2 \) for \( t = 0 \) by considering \( \alpha = 0.5, \gamma = 0.5, \kappa = 0.5, \omega = 2, \lambda = 0.75, \theta = -5 \) and \( K = 0 \).
with \((\gamma \kappa^3 + \alpha \kappa^2 + \omega)(\gamma \lambda (6 \gamma \kappa + 2 \alpha)) > 0\) for the presence of solutions.

\[
\begin{align*}
&\ u_{2,3}(x,t) = -\frac{\sqrt{\gamma \kappa^3 + \alpha \kappa^2 + \omega}}{\gamma \lambda (6 \gamma \kappa + 2 \alpha)} \sqrt{2} \tanh \left( \sqrt{\frac{\gamma \kappa^3 + \alpha \kappa^2 + \omega}{6 \gamma \kappa + 2 \alpha}} \left( x - \left( \frac{\gamma \kappa^3 + \gamma \alpha \kappa^2 + \gamma \omega}{3 \gamma \kappa + \alpha} - 2 \alpha \kappa - 3 \gamma \kappa^2 \right) t + K \right) \right) , \\
\end{align*}
\]

\[
\begin{align*}
&\ u_{4,3}(x,t) = -\frac{\sqrt{\gamma \kappa^3 + \alpha \kappa^2 + \omega}}{\gamma \lambda (6 \gamma \kappa + 2 \alpha)} \sqrt{2} \coth \left( \sqrt{\frac{\gamma \kappa^3 + \alpha \kappa^2 + \omega}{6 \gamma \kappa + 2 \alpha}} \left( x - \left( \frac{\gamma \kappa^3 + \gamma \alpha \kappa^2 + \gamma \omega}{3 \gamma \kappa + \alpha} - 2 \alpha \kappa - 3 \gamma \kappa^2 \right) t + K \right) \right) , \\
\end{align*}
\]

with \((\gamma \kappa^3 + \alpha \kappa^2 + \omega)(\gamma \lambda (6 \gamma \kappa + 2 \alpha)) < 0\) for the presence of solutions. (Fig. 6).

Set 3:

\[
a_2 = \pm \frac{\sqrt{2 \gamma \lambda}}{2 \gamma \lambda} , a_1 = 0 , \ a_3 = 0 , \ a_3 = 0 .
\]

Substituting these parameters into Eqs. (2.8) and (2.14), we obtained the following solution that satisfies the Eq. (1.2):

\[
\begin{align*}
&\ u_{3,3}(x,t) = -\frac{\sqrt{2 \gamma \lambda (\kappa t + \kappa - 2 \omega + 3 \kappa)}}{\gamma \lambda (6 \gamma \kappa + 2 \alpha) K} , \\
\end{align*}
\]

with \(\gamma \lambda > 0\) for the presence of solutions (Fig. 7).

4. Conclusion

In this study, the perturbed RKL model and its special case which is called a generalized RKL model with Kerr law nonlinearity are taken into account. In order to get some of the exact solutions of these nonlinear complex models, the Riccati-Bernoulli sub-ODE technique is carried out. Besides, all obtained soliton solutions that satisfy the corresponding main equation are successfully produced for these models. To demonstrate the behavior of the acquired solitons, we illustrate some 3D and 2D graphs through suitable parameter values. Consequently, it can be figured out that the utilized method is a powerful mathematical technique for complex nonlinear models.
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