The Schrödinger-KdV equation of fractional order with Mittag-Leffler nonsingular kernel

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Abstract Fractional order differential equations are utilized for modeling many complicated physical and natural phenomena in nonlinear sciences and related fields. In this manuscript, the fractional order Schrödinger-KdV equation in the sense of Atangana-Baleanu derivative is investigated. The Schrödinger-KdV equation demonstrates various types of wave propagation such as Langmuir wave, dust-acoustic wave and electromagnetic waves in plasma physics. Using the fixed-point theorem, the existence and uniqueness to the solution of the studied nonlinear model is established. Using the modified Laplace decomposition method, we establish the exact solution to fractional order Schrödinger-KdV equation. The numerical simulations to the reported result are presented. The comparison between analytical and numerical approximations is also presented. It is shown that the approximate-analytical results are compatible with the analytical results via the $L_2$ and $L_1$ error norms. We compare our result with some existing results in the literature.

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1. Introduction

A certain relation between an unknown function and its partial derivatives can be enunciated by partial differential equations (PDEs). In all areas of engineering and sciences, PDEs can frequently be found. The use of PDEs in areas such as, biology, finance, image processing and graphics, and social sciences...
has increased drastically in recent years. As a consequence, when some independent variables interact with each other in each of the aforementioned field, a suitable functions in these variables can be defined and thereby modeling a multifarious processes by establishing equations for the associated functions. There are several facets of study of PDEs. The classical approach which dominated the 19th century was to establish techniques for discovering explicit solutions. The theoretical analysis of PDEs has many applications. It should be emphasized that very complex equations exist which cannot be solved even by means of supercomputers. In these cases, all one can do is try to get qualitative information on the solution. Furthermore, the formulation of the equation and its related side conditions is a critically important issue. The equation usually derives from a model of a physical or engineering problem. The fact that the model is indeed consistent in the sense that it leads to a solvable PDE is not immediately apparent. Moreover, in most cases it is desirable that the solution be unique, and that it be stable under small data disturbances. A theoretical understanding of the equation helps one to verify if these conditions are met [1–7]. Several techniques to solve classical PDEs have been proposed and many different solutions have been found [8–17].

Nonlinear Schrödinger equation (NLSE) is one form of PDEs whose research has achieved some remarkable success for decades, owing to its broad application range. Different forms of NLSE are used in various domains to describe particular phenomena, such as nonlinear optics [18–20], Bose–Einstein Delicacy [21,22], and fluid dynamics [23], as well as several others. The distinctive aspect between the types of NLSE found is some kind of non-linearity to generate more light on the disruption that occurs when the electromagnetic pulses propagate in the optical extreme [24,25]. In addition, current study differs from other studies that have been performed about the Schro¨ dinger-KdV equation in terms of several others. The distinctive aspect between the types of fractional operators have not been explored even by means of supercomputers. In these cases, all one can do is try to get qualitative information on the solution. Furthermore, the formulation of the equation and its related side conditions is a critically important issue. The equation usually derives from a model of a physical or engineering problem. The fact that the model is indeed consistent in the sense that it leads to a solvable PDE is not immediately apparent. Moreover, in most cases it is desirable that the solution be unique, and that it be stable under small data disturbances. A theoretical understanding of the equation helps one to verify if these conditions are met [1–7]. Several techniques to solve classical PDEs have been proposed and many different solutions have been found [8–17].

In order to give more analysis and define the solution methods, we will present some illustrative definitions about Atangana-Baleanu fractional derivative which has essential advantages when the problem is considered along with the initial conditions.

Definition 1. The Atangana-Baleanu (ABC) derivative considered in the Caputo operator mean is given by [58]

\[
\text{ABC}_{t}^{\alpha}D_{t}^{\gamma}\{g(t)\} = \frac{F(\alpha)}{1-\sigma} \int_{a}^{t} g(k) E_{\sigma}^{-\alpha} \left[ -\frac{\sigma}{1-\sigma} (t-k)^{\sigma} \right] dk, \tag{3}
\]

where \(F(\alpha)\) is a normalization function such that \(F(0) = F(1) = 1\), \(g \in H^{1}(a,b), b > a, \sigma \in [0,1]\) and \(E_{\sigma}\) represents the Mittag–Leffler function.

Definition 2. Suppose \(g \in H^{1}(a,b), b > a, \sigma \in [0,1]\) and non-differentiable function, then the Atangana-Baleanu (ABR) derivative in the Riemann–Liouville sense is presented by [58]

\[
\text{ABR}_{t}^{\alpha}D_{t}^{\gamma}\{g(t)\} = \frac{F(\alpha)}{1-\sigma} \frac{d}{dt} \int_{a}^{t} g(k) E_{\sigma}^{-\alpha} \left[ -\frac{\sigma}{1-\sigma} (t-k)^{\alpha} \right] dk. \tag{4}
\]

Definition 3. The LT associated with the AB derivative is given by [60]

\[
\mathcal{L}\{\text{ABR}_{t}^{\alpha}D_{t}^{\gamma}\{g(t)\}\}(s) = \frac{F(\alpha) s^{\alpha} \mathcal{L}\{g(t)\}(s) - s^{\alpha-1} g(0)}{(1-\sigma)(s^{\alpha} + \frac{\sigma}{1-\sigma})}. \tag{5}
\]

Definition 4. The Atangana-Baleanu fractional integral operator of order \(\sigma\) is given as [60]

\[
\text{ABC}_{t}^{-\alpha}g(t) = \frac{1-\sigma}{F(\alpha)} g(t) + \frac{\sigma}{F(\alpha)\Gamma(\sigma)} \int_{a}^{t} g(k)(t-k)^{\sigma-1} dk, \tag{6}
\]

where \(0 < \sigma < 1\), and \(g\) is a function of \(\sigma\), and \(\Gamma(\sigma)\) is the Euler Gamma function.
3. Qualitative Properties of the Nonlinear Schrödinger-KdV Model

In this section, using the fixed-point theory, we present some qualitative properties of the studied nonlinear model.

Consider the fractional KdV-Schrödinger equation with ABC derivative

\[ i^{ABC}D_t^s p = p_{xx} + pq, \]

\[ ABC_y q = -6aq_x - q_{xxx} + \left( |p|^2 \right)_x, \]  

subject to the initial conditions

\[ u(x,0) = \varphi_1(x), \quad v(x,0) = \varphi_2(x). \]  

Considering the fact that \( p = u + iv \) and taking \( h = 0 \) in Eq. (7), one can construct the following system

\[ ABC_y u - v_{xx} - vq = 0, \]

\[ ABC_y v + u_{xx} + uq = 0, \]

\[ ABC_y q + 6aq_x + q_{xxx} - 2mu_x - 2vv_x = 0. \]  

Applying the integral operator of the AB derivative given in Eq. (6) to the system (7), we have the following integral equations

\[ p(x,t) - p(x,0) = (1-\sigma)G_1(x,t,p) + \frac{\sigma}{F(\sigma)\Gamma(\sigma)} \int_0^t (t-k)^{\sigma-1} G_1(x,k,p)dk, \]  

and

\[ q(x,t) - q(x,0) = (1-\sigma)G_2(x,t,q) + \frac{\sigma}{F(\sigma)\Gamma(\sigma)} \int_0^t (t-k)^{\sigma-1} G_2(x,k,q)dk. \]  

Theorem 1. Suppose that \( p(x,t) \) and \( q(x,t) \) are bounded functions, then the operators \( \Psi(p(x,t)) \) and \( \Psi(q(x,t)) \) expressed as

\[ \Psi(p(x,t)) = p(x,0) + \frac{(1-\sigma)}{F(\sigma)}G_1(x,t,p) \]

\[ + \frac{\sigma}{F(\sigma)\Gamma(\sigma)} \int_0^t (t-k)^{\sigma-1} G_1(x,k,p)dk, \]  

and

\[ \Psi(q(x,t)) = q(x,0) + \frac{(1-\sigma)}{F(\sigma)}G_2(x,t,q) \]

\[ + \frac{\sigma}{F(\sigma)\Gamma(\sigma)} \int_0^t (t-k)^{\sigma-1} G_2(x,k,q)dk, \]  

respectively, satisfy the LC.

Proof: Suppose that \( p(x,t) \) and \( p_1(x,t) \) are bounded, then

\[ \| \Psi(p(x,t)) - \Psi(p_1(x,t)) \| = \| \left( \frac{(1-\sigma)}{F(\sigma)}G_1(x,t,p) - G_1(x,t,p_1) \right) \| + \frac{\sigma}{F(\sigma)\Gamma(\sigma)} \int_0^t (t-k)^{\sigma-1} \| G_1(x,k,p) - G_1(x,k,p_1) \| dk \]

\[ \leq \left( \frac{(1-\sigma)}{F(\sigma)} \| G_1(x,t,p) - G_1 \| \right) \]

\[ \left( \frac{t}{\Gamma(\sigma)} \| (t-k)^{\sigma-1} \| G_1(x,k,p) - G_1(x,k,p_1) \| dk \right) \]

\[ \leq \left( \frac{(1-\sigma)}{F(\sigma)} \frac{\sigma}{F(\sigma)\Gamma(\sigma)} \| (t-k)^{\sigma-1} \| \right) \| p - p_1 \|. \]

Thus,

\[ \| \Psi(p(x,t)) - \Psi(p_1(x,t)) \| \leq \Delta_1 \| p - p_1 \|, \]  

where \( \Delta_1 = \left( \frac{(1-\sigma)}{F(\sigma)} \frac{\sigma}{F(\sigma)\Gamma(\sigma)} \| (t-k)^{\sigma-1} \| \right) \). Similarly, assuming \( q(x,t) \) and \( q_1(x,t) \) to be bounded functions, we have

\[ \| \Psi(q(x,t)) - \Psi(q_1(x,t)) \| \leq \Delta_2 \| q - q_1 \|. \]  

Thus, the operators \( \Psi(p(x,t)) \) and \( \Psi(q(x,t)) \) satisfy the Lipschitz condition.

Theorem 2. Suppose that \( q(x,t) \) and \( p(x,t) \) are bounded functions, then the operators given by

\[ \Theta(p) = -i(p_{xx} + pq) \]  

and

\[ \Theta(q) = -6aq_x - q_{xxx} + \left( |p|^2 \right)_x \]  

satisfy the conditions

\[ \| \Theta(p) - \Theta(p_1) \| \leq \xi_1 \| p - p_1 \|, \]  

and

\[ \| \Theta(q) - \Theta(q_1) \| \leq \xi_2 \| q - q_1 \|, \]  

respectively [59].

Proof: Suppose that \( p(x,t) \) and \( q(x,t) \) are bounded functions, then

\[ \| \Theta(p) - \Theta(p_1) \| = \| -i(p_{xx} - p_{xxx}) + (p - p_1)q \| \]

\[ \leq \| -i(p - p_1)_{xx} + (p - p_1)q \| \]

\[ \leq \| (p - p_1)_{xx} \| \| p - p_1 \| + \| iq \| \| p - p_1 \|^2 \leq \xi_1 \| p - p_1 \|^2. \]

Hence,

\[ \| \Theta(p) - \Theta(p_1) \| \leq \xi_1 \| p - p_1 \|, \]  

Similarly, for the unknown function \( q(x,t) \):

\[ \| \Theta(q) - \Theta(q_1) \| \leq \xi_2 \| q - q_1 \|. \]  

Therefore, this gives the proof.

Theorem 3. Assume that \( p(x,t) \) and \( q(x,t) \) are bounded functions and \( 0 < \| \mathcal{M} \| < \infty \), then the operators

\[ \Theta(p) = -i(p_{xx} + pq) \]  

(22)
and

\[ \Theta(q) = -6qy_x - q_{xxx} + \left( |p|^2 \right)_x \]  

(23)

satisfy the conditions

\[ |(\Theta(p) - \Theta(p_1), \mathcal{H})| \leq \xi_1 |p - p_1|||\mathcal{H}| | \]  

(24)

and

\[ |(\Theta(q) - \Theta(q_1), \mathcal{H})| \leq \xi_2 |q - q_1|||\mathcal{H}| | \]  

(25)

respectively [59].

**Proof:** Suppose that \( p(x, t) \) and \( q(x, t) \) are bounded functions and \( 0 < ||\mathcal{H}|| < \infty \), then

\[ |(\Theta(p) - \Theta(p_1), \mathcal{H})| = |\langle (p_{xxx} - p_{xxx}) + (\rho - \rho_1)q, \mathcal{H} \rangle | \leq \xi_1 |p - p_1|||\mathcal{H}| | \]  

(26)

Similarly, for the unknown function \( q(x, t) \):

\[ |(\Theta(q) - \Theta(q_1), \mathcal{H})| \leq \xi_2 |q - q_1|||\mathcal{H}| | \]  

(27)

These results give the proof. \( \square \)

**4. Existence and Uniqueness of the Solution**

In this section, we present the existence and uniqueness of Eq. (7). Taking the unknown functions into account, \( p(x, t) \) and \( q(x, t) \), the following iterative formulas are constructed:

\[ p_{n+1}(x, t) = \frac{(1 - \sigma)}{F(\sigma)} G_1(x, t, p_n) + \frac{\sigma}{F(\sigma) \Gamma(\sigma)} \int_0^t (t - k)^{\sigma - 1} G_1(x, k, p_n) dk, \]  

(28)

\[ q_{n+1}(x, t) = \frac{(1 - \sigma)}{F(\sigma)} G_2(x, t, q_n) + \frac{\sigma}{F(\sigma) \Gamma(\sigma)} \int_0^t (t - k)^{\sigma - 1} G_2(x, k, q_n) dk, \]  

(29)

\[ p_0 = p(x, 0) \quad \text{and} \quad q_0 = q(x, 0). \]

The sequential terms can be given as

\[ r_n(x, t) = p_n(x, t) - p_{n-1}(x, t) \]

\[ = \frac{(1 - \sigma)}{F(\sigma)} \left( G_1(x, t, p_{n-1}) - G_1(x, t, p_{n-2}) \right) \]

\[ + \frac{\sigma}{F(\sigma) \Gamma(\sigma)} \int_0^t (t - k)^{\sigma - 1} \left( G_1(x, k, p_{n-1}) - G_1(x, k, p_{n-2}) \right) dk, \]  

(30)

and

\[ \psi_n(x, t) = q_n(x, t) - q_{n-1}(x, t) \]

\[ = \frac{(1 - \sigma)}{F(\sigma)} \left( G_2(x, t, q_{n-1}) - G_2(x, t, q_{n-2}) \right) \]

\[ + \frac{\sigma}{F(\sigma) \Gamma(\sigma)} \int_0^t (t - k)^{\sigma - 1} \left( G_2(x, k, q_{n-1}) - G_2(x, k, q_{n-2}) \right) dk, \]  

(31)

where

\[ p_n(x, t) = \sum_{\gamma=0}^{n} \zeta_1(x, t), \]  

(32)

\[ q_n(x, t) = \sum_{\gamma=0}^{n} \psi_1(x, t). \]  

(33)

Considering the norm of both sides of Eq. (30), produces

\[ ||r_n(x, t)|| = ||p_n(x, t) - p_{n-1}(x, t)|| \]

\[ = \left| \frac{(1 - \sigma)}{F(\sigma)} \left( G_1(x, t, p_{n-1}) - G_1(x, t, p_{n-2}) \right) \right| \]

\[ + \frac{\sigma}{F(\sigma) \Gamma(\sigma)} \int_0^t (t - k)^{\sigma - 1} \left( G_1(x, k, p_{n-1}) - G_1(x, k, p_{n-2}) \right) dk \]  

(34)

By triangular inequality, one may see that

\[ ||r_n(x, t)|| \leq \frac{1 - \sigma}{F(\sigma)} ||p_{n-1} - p_{n-2}|| \]

\[ + \frac{\sigma}{F(\sigma) \Gamma(\sigma)} \int_0^t (t - k)^{\sigma - 1} ||p_{n-1} - p_{n-2}|| dk. \]  

(35)

This is because the kernel fulfills the LC, we get

\[ ||r_n(x, t)|| \leq \frac{1 - \sigma}{F(\sigma)} \xi_1 ||p_{n-1} - p_{n-2}|| \]

\[ + \frac{\sigma}{F(\sigma) \Gamma(\sigma)} \xi_1 \int_0^t (t - k)^{\sigma - 1} ||p_{n-1} - p_{n-2}|| dk. \]  

(36)

By following the similar steps, one can conclude that \( q(x, t) \) has the following form,

\[ ||\psi_n(x, t)|| \leq \frac{1 - \sigma}{F(\sigma)} \xi_2 ||q_{n-1} - q_{n-2}|| \]

\[ + \frac{\sigma}{F(\sigma) \Gamma(\sigma)} \xi_2 \int_0^t (t - k)^{\sigma - 1} ||q_{n-1} - q_{n-2}|| dk. \]  

(37)

Putting Eqs. (28)-(37), the following theorem is constructed:

**Theorem 4.** Suppose that \( p(x, t) \) and \( q(x, t) \) are bounded functions, then equation (Eq. (7)) is said to have a solution if there exists \( t_0 \) and the following inequality holds [59]:

\[ \left( \frac{1 - \sigma}{F(\sigma)} \right)^2 \xi_1^2 + \frac{\sigma}{F(\sigma) \Gamma(\sigma)} \xi_1 t_0 \leq 1. \]  

(38)

**Proof:** Suppose that \( p(x, t) \) is a bounded function. Putting Eq. (36) into consideration, and using the recursive scheme, we have

\[ ||r_n(x, t)|| \leq \left( \frac{1 - \sigma}{F(\sigma)} \right)^n \xi_1^2 + \frac{\sigma}{F(\sigma) \Gamma(\sigma)} \xi_1 t_0 \]  

(39)

Hence, Eq. (32) exists and its smooth. We now show that Eq. (32) is a solution to Eq. (7).

Assume that \( p(x, t) - p(x, 0) = p_n(x, t) - r_n(x, t) \), then we have

\[ ||r_n(x, t)|| = ||\frac{(1 - \sigma)}{F(\sigma)} (G_1(x, t, p) - G_1(x, t, p_{n-1})) \]

\[ + \frac{\sigma}{F(\sigma) \Gamma(\sigma)} \int_0^t (t - k)^{\sigma - 1} (G_1(x, k, p) - G_1(x, k, p_{n-1})) dk \]  

(40)

\[ \leq \frac{(1 - \sigma)}{F(\sigma)} \xi_1 ||p - p_{n-1}|| + \frac{\sigma}{F(\sigma) \Gamma(\sigma)} \xi_1 t_0 ||p - p_{n-1}||. \]

Using the recursive relation, we have

\[ ||r_n(x, t)|| \leq \left( \frac{1 - \sigma}{F(\sigma)} + \frac{1}{F(\sigma) \Gamma(\sigma)} t_0 \right)^{n+1} \xi_1 t_0 \]  

(41)

at \( t = t_0 \), Eq. (41) becomes
as \( n \to 0 \) implies that \( \|r_n(x, t)\| \to 0 \).

Similarly, for \( q(x, t) \),
\[
\|d_n(x, t)\| = \left[ \frac{(1 - \sigma)}{F(\sigma)} + \frac{1}{F(\sigma)\Gamma(\sigma)} g_0^\delta \right]^{n+1} \xi_1^n\delta_n.
\]
(43)
as \( n \to 0 \) implies that \( \|d_n(x, t)\| \to 0 \).

Suppose now that Eq. (7) has two solutions, \( p(x, t) \) and \( u(x, t) \), then
\[
p(x, t) - u(x, t) = \frac{1}{F(\sigma)} \left[ (G_1(x, t, p) - G_1(x, t, u)) + \frac{\sigma}{F(\sigma)} \int_k \left( t-k \right)^{\nu-1} \left( G_1(x, k, p) - G_1(x, k, u) \right) dk \right].
\]
(44)
Taking norm of both sides in Eq. (44), we get
\[
||p(x, t) - u(x, t)|| = \left[ \frac{1}{F(\sigma)} \left| (G_1(x, t, p) - G_1(x, t, u)) + \frac{\sigma}{F(\sigma)} \int_k \left( t-k \right)^{\nu-1} \left( G_1(x, k, p) - G_1(x, k, u) \right) dk \right| \right].
\]
(45)
Hence,
\[
\left| p(x, t) - u(x, t) \right| \leq \left| \frac{(1 - \sigma)}{F(\sigma)} \xi_1^\delta \right| ||p(x, t) - u(x, t)|| + \frac{\sigma}{F(\sigma)} \xi_1^\delta ||p(x, t) - u(x, t)||.
\]
(46)
Thus,
\[
||p(x, t) - u(x, t)|| \leq \left( 1 - \frac{(1 - \sigma)}{F(\sigma)} \xi_1^\delta - \frac{1}{F(\sigma)\Gamma(\sigma)} \xi_1^\delta \right) \leq 0.
\]
(47)
By following the similar steps, one can conclude that \( q(x, t) \) has the following form,
\[
\left| q(x, t) - v(x, t) \right| \leq \left( 1 - \frac{(1 - \sigma)}{F(\sigma)} \xi_2 - \frac{1}{F(\sigma)\Gamma(\sigma)} \xi_1^\delta \right) \leq 0.
\]
(48)

**Theorem 5.** The fractional nonlinear Schrödinger equation (Eq. (7)) is said to have a unique solution if [59]
\[
\left( 1 - \frac{(1 - \sigma)}{F(\sigma)} \xi_1 - \frac{1}{F(\sigma)\Gamma(\sigma)} \xi_1^\delta \right) > 0.
\]
(49)

**Proof:** By considering the conditions given by Theorem 4, then we have
\[
||p(x, t) - u(x, t)|| \leq \left( 1 - \frac{(1 - \sigma)}{F(\sigma)} \xi_1 - \frac{1}{F(\sigma)\Gamma(\sigma)} \xi_1^\delta \right) \leq 0.
\]
(50)
It implies that
\[
||p(x, t) - u(x, t)|| = 0,
\]
(51)
that results \( p(x, t) = u(x, t) \).

For \( q(x, t) \), we can have that \( q(x, t) = v(x, t) \) taking into account the similar way. This fact concludes that Eq. (7) has a unique solution. □

5. Modified Laplace Decomposition Method

This section defines the solution method by using the Atangana-Baleanu operator and the behaviors of the solutions according to this method are presented. Here we propose an infinite series solution method namely modified Laplace decomposition method (MLDM). Firstly, we consider the KdV-Schrödinger equation which is defined in Eq. (9), where the initial condition is given in the following system [57]
\[
u_0 = u(x, 0) = \tanh(x)\cos(x),
\]
\[
u_0 = v(x, 0) = \tanh(x)\sin(x),
\]
\[
u_0 = q(x, 0) = 7/8 - 2\tanh^2(x),
\]
(52)
as the following form:
\[
\frac{\partial}{\partial x} \left( \frac{\partial^\nu}{\partial t^\nu} u(x, t) + F^1\left[ u(x, t) \right] + H^1\left[ u(x, t) \right] \right) = K^1(x, t),
\]
\[
\frac{\partial}{\partial x} \left( \frac{\partial^\nu}{\partial t^\nu} v(x, t) + F^2\left[ v(x, t) \right] + H^2\left[ v(x, t) \right] \right) = K^2(x, t),
\]
\[
\frac{\partial}{\partial x} \left( \frac{\partial^\nu}{\partial t^\nu} q(x, t) + F^3\left[ q(x, t) \right] + H^3\left[ q(x, t) \right] \right) = K^3(x, t).
\]
(53)
In Eq. (53) we explain with the function \( F^1\left[ v(x, t) \right] \) the linear part, with the function \( H^1\left[ v(x, t) \right] \) the nonlinear term and with the function \( K^1\left[ v(x, t) \right] \) the nonhomogeneous part, where \( \nu = 1, 2, 3 \) for the functions \( u, v \) and \( q \), respectively. Using the Laplace transform of the AB operator which is given in Definition 3, we define the \( \mathcal{L}\{u(x, t)\} = \mathcal{L}\{\tilde{U}(x, \kappa)\} \), \( \mathcal{L}\{v(x, t)\} = \mathcal{L}\{\tilde{V}(x, \kappa)\} \), and \( \mathcal{L}\{q(x, t)\} = \mathcal{L}\{\tilde{Q}(x, \kappa)\} \). Then, the suggested method provides the following-type a series solution:
\[
u(x, t) = \sum_{j=0}^{\infty} \nu_j(x, t).
\]
(54)
In Eq. (54), the nonlinear terms shown by \( H[u(x, t)] \) are derived as
\[
H[u] = \sum_{j=0}^{\infty} \mathcal{L}\{u_0, u_1, \ldots, u_j\},
\]
(55)
in which \( \mathcal{L}_j \) is the Adomian polynomial which can be obtained by [61]
\[
\mathcal{L}_j(u_0, u_1, \ldots, u_j) = \frac{1}{j!} \int_0^1 \frac{d^j}{d\kappa^j} H \left[ \sum_{\ell=0}^{\infty} \nu^{\ell} \right] \bigg|_{\kappa=0}, \quad j \geq 0.
\]
(56)
Corresponding Adomian polynomials of the proposed nonlinear sections can be easily computed by setting a suitable code. In what follows, we give the modified decomposition elements constructed with the AB operator for Eq. (53). Taking the Laplace transform of both sides of Eq. (53), we have for the first equation of the system
\[
\tilde{U}(x, \kappa) = \frac{1}{\kappa} \left[ u(x, 0) - \frac{\sigma + (1 - \sigma)\kappa^\nu}{\kappa^\nu} \left( \mathcal{L}\{F^1[u]\} + \mathcal{L}\{H^1[u]\} - \mathcal{L}\{K^1(x, t)\} \right) \right],
\]
(57)
which can also be adopted by considering the same way for the functions \( v \) and \( q \). Then we apply the inverse LT to Eq. (57), we have
\[
u(x, t) = Y^1(x, t)
\]
\[
= \mathcal{L}^{-1} \left\{ \frac{\sigma + (1 - \sigma)\kappa^\nu}{\kappa^\nu} \left( \mathcal{L}\{F^1[u]\} + \mathcal{L}\{H^1[u]\} \right) \right\}.
\]
(58)
where \( Y^1(x, t) = u(x, 0) + \mathcal{L}^{-1} \left\{ \frac{\sigma + (1 - \sigma)\kappa^\nu}{\kappa^\nu} \mathcal{L}\{K^1(x, t)\} \right\} \). If the term \( Y^1(x, t) \) can be assumed as \( Y^1(x, t) = Y^1_1(x, t) + Y^1_2(x, t) \),
then one can construct the recursive algorithm for the first component $u_0(x, t)$ and the second one $u_1(x, t)$ in terms of the modified Laplace decomposition method (MLDM) as

$$u_0(x, t) = Y_1^0(x, t),$$  \hspace{1cm} (59)

and

$$u_1(x, t) = Y_1^1(x, t) - \mathcal{L}^{-1}\left\{\frac{\sigma + (1 - \sigma)\kappa^2}{\kappa^2} \left[\mathcal{L}\left\{F^\sigma[u]\right\} + \mathcal{L}\left\{H^\sigma[u]\right\}\right]\right\},$$  \hspace{1cm} (60)

respectively. Consequently, the relationship between the iterations gets

$$u_{m+1}(x, t) = -\mathcal{L}^{-1}\left\{\frac{\sigma + (1 - \sigma)\kappa^2}{\kappa^2} \left[\mathcal{L}\left\{F^\sigma[u_m(x, t)]\right\}\right]\right\}$$

$$- \mathcal{L}^{-1}\left\{\frac{\sigma + (1 - \sigma)\kappa^2}{\kappa^2} \mathcal{L}\left\{H^\sigma[u_m(x, t)]\right\}\right\}. \hspace{1cm} (61)$$

Therefore, it can be approximated the solution $u(x, t)$ by considering the series $u(x, t) = \sum_{m=0}^{\infty} u_m(x, t)$. For the functions $r$ and $q$, approximate solutions are given as

$$r(x, t) = \sum_{m=0}^{\infty} r_m(x, t),$$

and

$$q(x, t) = \sum_{m=0}^{\infty} q_m(x, t),$$

respectively. For the functions $\psi(x, t)$ and $\varphi(x, t)$, respectively.

5.1. $L_2$ and $L_\infty$ Error Norms of the Method

In this section, as a further step we have achieved numerical error values by using $L_2$ and $L_\infty$ error norms to point out how the approximate-analytical solutions are compatible with the analytical results. Then the $L_2$ error norm can be given as [62]

$$L_2 = \left\|Y_{\text{exact}} - Y_{\text{numeric}}\right\|_2 = \sqrt{\Delta t \sum_{k=0}^{N} \left|Y_{k \text{exact}} - Y_{k \text{numeric}}\right|^2}, \hspace{1cm} (62)$$

and also $L_\infty$ error norm

$$L_\infty = \left\|Y_{\text{exact}} - Y_{\text{numeric}}\right\|_\infty = \max_k \left|Y_{k \text{exact}} - Y_{k \text{numeric}}\right|. \hspace{1cm} (63)$$

6. Further Results and Discussion

In this section, we give the fundamental solution to the mentioned problem. We also present the characterization of the solutions by considering their graphical representations and table structures. Then we construct the following iterations:

$$u_0(x, t) = \tanh(x) \cos(t),$$
$$u_1(x, t) = \tanh(x) \sin(t),$$
$$q_0(x, t) = \frac{7}{8} - 2\tanh^2(x),$$
$$u_1(x, t) = \frac{\sin(x) \tanh(x) \left[-2\tanh^2(x)\right] \left(1 + \tanh^2(x)\right)}{1 + \tanh^2(x)} + \ldots,$$
$$v_1(x, t) = -\frac{\cos(x) \tanh(x) \left[2\tanh^2(x)\right] \left(1 + \tanh^2(x)\right)}{1 + \tanh^2(x)} + \ldots,$$
$$q_1(x, t) = \frac{1 + \tanh^2(x)}{1 + \tanh^2(x)} + \frac{1 + \tanh^2(x)}{1 + \tanh^2(x)} + \ldots,$$

Continuing like this, one can obtain the other components of the series. Then the solution is given as it has been explained in Section 5.

In the following Figs. 1–3, we have pointed out the comparison of the solutions obtained by numerical solutions and the exact solutions given by [57]. According to the results, one can conclude that the method gives the numerical solution around the exact solution and the method is fully agreement with the exact solution results. Our work differs from other studies that have been performed about the Schrödinger-KdV equation in terms of it has pointed out the accurate numerical solution with a series scheme that has been constructed with a non-singular kernel operator. On the other hand, the scheme that has been defined in Section 5 identifies the components of the series solution. It is possible to calculate more components in the scheme to increase the approximation to the exact solution. In this context, we have used only first four components to approximate the exact solution and to generate the surfaces shown in all figures in this paper. It can be considered as a major advantage of the solution method to obtain the solutions even in fewer terms. Another advantage of the method is to simplify the calculations by avoiding the difficulties and massive computational work compared with traditional numerical methods, because the MLDM appears

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**Fig. 1** Comparing the numerical and exact $u$-solutions in Eq. (9) for various values of $x$ and $t$. 
to be very promising for solving nonlinear partial differential equations or systems without linearization, perturbation, or discretization [27].

In Tables 1–3, we have shown that the series solution results and comparison of the solution with the exact solution. Moreover, we have presented the absolute error values which are in a good range.

In Table 4, we have presented $L_2$ and $L_{\infty}$ error norms that have been explained in Section 5.1 for each solution of the problem given in Eq. (9).

6.1. Comparative analysis

In this subsection, we have presented some researches made in the literature previously and in relation to our paper. With this comparative analysis, we aim to show that our paper has a great impact on obtaining numerical results of the Schrödinger equation of fractional order and to point out the accuracy and effectiveness of the method we have considered. Moreover, we aim to represent the advantages of the ABC fractional order

Table 1 Numerical and exact solutions with absolute errors of $u$-solutions in Eq. (9) for various values of $x$ and $t$.

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>$t_j$</th>
<th>MLSM Solution</th>
<th>Exact Solution</th>
<th>Absolute Errors</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>$10^{-2}$</td>
<td>0.0199931510416666</td>
<td>0.019992818906816</td>
<td>$3.32134 \times 10^{-7}$</td>
</tr>
<tr>
<td>0.1</td>
<td>$10^{-2}$</td>
<td>0.0299766867695441</td>
<td>0.029976360385966</td>
<td>$3.27273 \times 10^{-7}$</td>
</tr>
<tr>
<td>0.2</td>
<td>$10^{-2}$</td>
<td>0.039944990402471</td>
<td>0.039944672022234</td>
<td>$3.18380 \times 10^{-7}$</td>
</tr>
<tr>
<td>0.3</td>
<td>$10^{-2}$</td>
<td>0.0498930850981510</td>
<td>0.049892779920108</td>
<td>$3.05178 \times 10^{-7}$</td>
</tr>
<tr>
<td>0.4</td>
<td>$10^{-2}$</td>
<td>0.0598160140592043</td>
<td>0.059815726655192</td>
<td>$2.87404 \times 10^{-7}$</td>
</tr>
<tr>
<td>0.5</td>
<td>$10^{-2}$</td>
<td>0.0697088401124785</td>
<td>0.069708575303005</td>
<td>$2.64809 \times 10^{-7}$</td>
</tr>
<tr>
<td>0.6</td>
<td>$10^{-2}$</td>
<td>0.0795666505949705</td>
<td>0.079566413433805</td>
<td>$2.37161 \times 10^{-7}$</td>
</tr>
</tbody>
</table>
Table 2 Numerical and exact solutions with absolute errors of $s$-solutions in Eq. (9) for various values of $x$ and $t$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$t$</th>
<th>MLSM Solution</th>
<th>Exact Solution</th>
<th>Absolute Errors</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>$10^{-2}$</td>
<td>0.000250000000</td>
<td>0.000250000000</td>
<td>0.000250000000</td>
</tr>
<tr>
<td>0.1</td>
<td>$10^{-2}$</td>
<td>0.00037244091072</td>
<td>0.00037244091072</td>
<td>0.00037244091072</td>
</tr>
<tr>
<td>0.2</td>
<td>$10^{-2}$</td>
<td>0.0016489301382071</td>
<td>0.0016489301382071</td>
<td>0.0016489301382071</td>
</tr>
<tr>
<td>0.3</td>
<td>$10^{-2}$</td>
<td>0.0025596317925977</td>
<td>0.0025596317925977</td>
<td>0.0025596317925977</td>
</tr>
<tr>
<td>0.4</td>
<td>$10^{-2}$</td>
<td>0.0036688051298385</td>
<td>0.0036688051298385</td>
<td>0.0036688051298385</td>
</tr>
<tr>
<td>0.5</td>
<td>$10^{-2}$</td>
<td>0.0049757876996172</td>
<td>0.0049757876996172</td>
<td>0.0049757876996172</td>
</tr>
<tr>
<td>0.6</td>
<td>$10^{-2}$</td>
<td>0.0064797996216566</td>
<td>0.0064797996216566</td>
<td>0.0064797996216566</td>
</tr>
</tbody>
</table>

Table 3 Numerical and exact solutions with absolute errors of $q$-solutions in Eq. (9) for various values of $x$ and $t$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$t$</th>
<th>MLSM Solution</th>
<th>Exact Solution</th>
<th>Absolute Errors</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.01</td>
<td>0.87396250000000000</td>
<td>0.87400213284987</td>
<td>0.000000000000</td>
</tr>
<tr>
<td>0.1</td>
<td>0.01</td>
<td>0.8728637422361050</td>
<td>0.873201079449458</td>
<td>0.000000000000</td>
</tr>
<tr>
<td>0.2</td>
<td>0.01</td>
<td>0.87136671869928667</td>
<td>0.871803410241155</td>
<td>0.000000000000</td>
</tr>
<tr>
<td>0.3</td>
<td>0.01</td>
<td>0.8694726241418140</td>
<td>0.869008321543135</td>
<td>0.000000000000</td>
</tr>
<tr>
<td>0.4</td>
<td>0.01</td>
<td>0.8671829682523867</td>
<td>0.866717244814800</td>
<td>0.000000000000</td>
</tr>
<tr>
<td>0.5</td>
<td>0.01</td>
<td>0.8644995231652466</td>
<td>0.86321924669360</td>
<td>0.000000000000</td>
</tr>
<tr>
<td>0.6</td>
<td>0.01</td>
<td>0.8614245703735080</td>
<td>0.86225445927326</td>
<td>0.000000000000</td>
</tr>
</tbody>
</table>

Table 4 $L_2$ and $L_{\infty}$ error norms of the method.

<table>
<thead>
<tr>
<th>$\Delta t$</th>
<th>MLSM $L_2$</th>
<th>MLSM $L_{\infty}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u(x,t)$</td>
<td>0.01</td>
<td>2.69239 x 10^{-6}</td>
</tr>
<tr>
<td>$v(x,t)$</td>
<td>0.01</td>
<td>1.47473 x 10^{-6}</td>
</tr>
<tr>
<td>$q(x,t)$</td>
<td>0.01</td>
<td>3.12834 x 10^{-3}</td>
</tr>
</tbody>
</table>

Conclusion

In this paper, we have considered the nonlinear time-fractional Schrödinger-KdV equation in the sense of Atangana-Baleanu derivative which can be able to do more extensive analysis due to the nonsingular kernel in its structure. This is because the Schrödinger-KdV equation demonstrates various types of wave propagation such as Langmuir, dust-acoustic and electromagnetic waves in plasma physics, it has been deeply attracting attention to the scientists in nonlinear sciences and related fields. For this reason, it is essential to obtain the solutions to the equation, to make mathematical analysis and to provide its qualitative properties. We have set the existence and uniqueness of the solution to the studied nonlinear model by considering the fixed-point theorem. We have achieved numerical solutions by using the modified Laplace decomposition method and have depicted the graphs of the solutions by comparing with the exact solutions. We have shown that the approximate-analytical results are fully compatible with the analytical results via the $L_2$ and $L_{\infty}$ error norms which have good approximations according the existing studies in the literature. We have also represented it by comparing our result with some existing results stated by the literature.

Declaration of Competing Interest

None.

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References


