New Soliton Solutions for the Higher-Dimensional Non-Local Ito Equation

6 authors, including:

Mustafa Inc
Firat University
555 PUBLICATIONS **7,816** CITATIONS

Emad A. Az-Zo'bi
Mu'tah University
28 PUBLICATIONS **222** CITATIONS

Adil Jhangeer
Namal Institute
80 PUBLICATIONS **544** CITATIONS

Some of the authors of this publication are also working on these related projects:

⬚ Articles, Books and posters for colleagues ⬚ View project

Computational and Applied Mathematics group on WhatsApp. View project

Mustafa Inc, E. A. Az-Zo'bi, Adil Jhangeer, Hadi Rezazadeh, Muhammad Nasir Ali, and
Mohammed K. A. Kaabar*

New Soliton Solutions for the Higher-Dimensional Non-Local Ito Equation

Abstract: In this article, (2+1)-dimensional Ito equation that models waves motion on shallow water surfaces is analyzed for exact analytic solutions. Two reliable techniques involving the simplest equation and modified simplest equation algorithms are utilized to find exact solutions of the considered equation involving bright solitons, singular periodic solitons, and singular bright solitons. These solutions are also described graphically while taking suitable values of free parameters. The applied algorithms are effective and convenient in handling the solution process for Ito equation that appears in many phenomena.

1 Introduction

Numerous complex phenomena that are encountered in mathematical physics, relativity, and economics are modeled via nonlinear differential equations [1]. There is no a single technique that can possibly solve different nonlinear evolution equations (NLEEs) kinds. Hence, several techniques have been introduced to construct exact solutions for NLEEs in the last few decades. Symbolic computational software programs can help interested researchers achieve these computational works. The obtained solutions allow scientists and engineers to understand the complex phenomena qualitative and measurable features in order to construct conclusions in an efficient way. As a result, various effective techniques have been proposed. $(1/G')$-expansion algorithm [2], bifurcation method [3], Hirota bilinear approach [4], sine-cosine method [5], Adomian decomposition algorithm and its

Mustafa Inc, Department of Computer Engineering, Biruni University, Istanbul, Turkey
Firat University, Science Faculty, Department of Mathematics, 23119 Elazig, Turkey
Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan, E-mail: minc@firat.edu.tr
E. A. Az-Zo'bi, Department of Mathematics and Statistics, Faculty of Science, Mutah University, Mutah, Al-Karak61710, P.O.Box 7, Jordan, E-mail: eaaz2006@yahoo.com
Adil Jhangeer, Department of Mathematics, Namal Institute, 30KM Talagang Road, Mianwali 42250, Pakistan, E-mail: adil.jhangeer@gmail.com
Hadi Rezazadeh, Faculty of Engineering Technology, Amol University of Special Modern Technologies, Amol, Iran, E-mail: rezazade-hadi1363@gmail.com
Muhammad Nasir Ali, Department of Sciences and Humanities, NUCES–FAST Lahore, Pakistan
Department of Mathematics, Govt. Islamia College Railway Road, Lahore, Pakistan, E-mail: taj467@gmail.com
*Corresponding Author: Mohammed K. A. Kaabar, Jabalia Camp, United Nations Relief and Works Agency (UNRWA) Palestinian Refugee Camp, Gaza Strip Jabalya, State of Palestine
Institute of Mathematical Sciences, Faculty of Science, University of Malaya, Kuala Lumpur 50603, Malaysia,
E-mail: mohammed.kaabar@wsu.edu

extensions [6–9], Exp-function technique [10], technique of F-expansion [11], He's variational iteration algorithm [12], inverse scattering approach [13], reduced differential transform method [14–17], new extended direct algebraic method [18], auxiliary equation method [19], first integral algorithm [20], residual power series method [21–25], simplest equation algorithm (SEM) [26, 27], modified simplest equation algorithm (MSEM)[28], and exp $(-\phi(\xi))$ method [29], that will be presented in the coming context, are examples of some attempts to solve such equations. The fractional calculus (FC) has started to be incredibly known in a few fields of science and engineering (see [55–61]). For the recent development with in this field in frame of the numerical scheme in the Caputo fractional order sense can easily be understood from recent published works [43–49]. For more information, readers can refer to some useful resources [50–54, 62–65] for deep understanding and learning about different aspects of this research area..

The SEM and its expansions [30–40] have constructed various NLEEs' solutions. With the help of SEM and MSEM, this article constructs novel exact solutions of

$$u_{tt} + u_{xxxt} + 3\left(2\,u_x u_t + u\,u_{xt}\right)$$
$$+ 3\,u_{xx}\left(\int u_t dx\right) + \alpha\,u_{yt} + \beta\,u_{xt} = 0, \qquad (1.1)$$

where the constants are denoted by α and β, and $u(x, y, t)$ represents the waves amplitude. By making use of differential operator $v = u_x$, Eq. (1.1) is going to be converted into the 5th-order PDE

$$v_{xtt} + v_{xxxxt} + 3\left(2\,v_{xx}v_{xt} + v_x\,v_{xxt}\right)$$
$$+ 3\,v_{xxx}v_t + \alpha\,v_{xyt} + \beta\,v_{xxt} = 0. \qquad (1.2)$$

The above Ito Eq. (1.2) (or equivalently Eq. (1.1) was initially established by a generalized bilinear KdV equation [41]. For $\alpha = \beta = 0$, we have the 1-D Ito equation. Researchers have recently analyzed the (2+1)-dimensional Ito equation. Wazwaz [42] implemented tanh–coth algorithm to obtain single soliton and periodic solutions. In this article, we have investigated the (2+1)-dimensional Ito equation with two different methods involving the simplest equation and modified simplest equation for exact analytic solutions.

This article is prepared as follows: The algorithms of SEM and MSEM are presented in section 2. By applying the proposed techniques, the Logarithmic form of obtained exact solutions for Eq. (1.1) is studied in section 3. Results' discussion including simulations of some obtained solutions is provided in section 4.

2 Mathematical Analysis

In this section, SEM and MSEM achieve the exact analytic solutions of constant-coefficients (2+1)-D PDE of the following form:

$$P(v, v_x, v_y, v_t, v_{xy} \ldots) = 0, \qquad (2.1)$$

where P is an assumed polynomial in $v = v(x, y, t)$ and its derivatives involving the highest derivative and the higher power of linear terms. The solution process is also valid for the time-dependent coefficients and NPDEs' systems.

To investigate Eq. (2.1), we transform it into an ODE under the transformation:

$$v(x, y, t) = v(\xi), \qquad (2.2)$$

where $\xi = x + y - \eta\,t$ is the wave variable and η is the wave frequency. In case of time-dependent, one can use $\eta = \eta(t)$. Under this consideration, Eq. (2.1) is reduced to be

$$F(v, v', v'', \ldots) = 0, \qquad (2.3)$$

where $v' = \frac{dv}{d\xi}$, $v'' = \frac{d^2 v}{d\xi^2}$, $v''' = \frac{d^3 v}{d\xi^3}$, …. By integrating Eq. (2.3) several times as possible as we can and setting the integration's constants to 0. The transformed equation is reduced and kept the solution process very simple.

From the above schemes, positive integer m is determined by balancing the highest order derivative and the nonlinear term of the highest order in the completely integrated form of Eq. (2.3). As a result, each method's solution process is discussed.

2.1 The Method of Simplest Equation

The Eq. (2.3) solution via the SEM [26, 27] is expressed as:

$$v(\xi) = \sum_{i=0}^{m} A_i\,\phi^i(\xi), \; A_m \neq 0, \qquad (2.4)$$

where A_i ($i = 0, 1, \ldots, m$) are parameters to be found later. $\phi(\xi)$ is the function that with the help of by the simplest equation can satisfy an ODE . The simplest equations in this article are the 1st order ODEs including the Bernoulli equation and Riccati equation. In the Bernoulli equation case, we have:

$$\phi'(\xi) = \lambda\,\phi(\xi) + \mu\,\phi^2(\xi). \qquad (2.5)$$

The solution function $\phi(\xi)$ is expressed as:

I. If $\lambda = 0$, we have the rational form:

$$\phi(\xi) = \frac{1}{\mu(\xi_0 - \xi)}. \tag{2.6}$$

II. If $\lambda > 0$ and $\mu < 0$, we have the rational-exponential form:

$$\phi(\xi) = \frac{\lambda\, e^{(\lambda(\xi+\xi_0))}}{1 - \mu\, e^{(\lambda(\xi+\xi_0))}}. \tag{2.7}$$

III. If $\lambda < 0$ and $\mu > 0$, we have the rational-exponential form:

$$\phi(\xi) = \frac{\lambda\, e^{(\lambda(\xi+\xi_0))}}{1 + \mu\, e^{(\lambda(\xi+\xi_0))}}. \tag{2.8}$$

Whereas in the case Riccati equation

$$\phi'(\xi) = \lambda\, \phi^2(\xi) + \mu, \tag{2.9}$$

and the solutions of Eq. (2.9) can be expressed in the following forms:

I. If $\lambda\mu < 0$, we have the hyperbolic form:

$$\phi(\xi) = -\frac{\sqrt{-\lambda\mu}}{\lambda} \tanh\left(\sqrt{-\lambda\mu}\,\xi - \frac{\varepsilon \ln \xi_0}{2}\right), \ \ \varepsilon = \pm 1, \tag{2.10}$$

or,

$$\phi(\xi) = -\frac{\sqrt{-\lambda\mu}}{\lambda} \coth\left(\sqrt{-\lambda\mu}\,\xi - \frac{\varepsilon \ln \xi_0}{2}\right). \tag{2.11}$$

II. If $\lambda\mu > 0$, we have the periodic form:

$$\phi(\xi) = \frac{\sqrt{\lambda\mu}}{\lambda} \tan\left(\sqrt{\lambda\mu}\,\xi + \xi_0\right), \tag{2.12}$$

or,

$$\phi(\xi) = -\frac{\sqrt{\lambda\mu}}{\lambda} \cot\left(\sqrt{\lambda\mu}\,\xi + \xi_0\right), \tag{2.13}$$

where ξ_0 is an integration constant.

From the Bernoulli equation given in Eq. (2.5), by the substitution of Eq. (2.4) into Eq. (2.3) and equating each coefficient with the same power in the constructed polynomial of $\phi(\xi)$ to 0, an algebraic equations' system in the variables μ, λ and the A_i's is constructed. This system is solved and substituted the determined values of μ, λ and the A_i's, associated with the Eq. (2.5) general solutions into Eq. (2.4). Then, the exact solution is obtained in a traveling-wave form for Eq. (2.1). By similarly doing this process again by the replacement of Eq. (2.5) by Eq. (2.9), new solutions' classes can be obtained. The scheme

of the simplest equation is applicable at the same time that the obtained system is solved in the undetermined parameters.

2.2 Modified Simple Equation Method

The MSEM [28] considers the Eq. (2.1) solution as follows:

$$v(\xi) = \sum_{i=0}^{m} A_i \left(\frac{\phi'(\xi)}{\phi(\xi)}\right)^i, \ \ A_m \neq 0, \tag{2.14}$$

where A_i ($i = 0, 1, \ldots, m$) are parameters to be determined later. Positive integer m is found by the homogeneous balance principle. $\phi(\xi)$ is an unspecified function to be determined later. Once Eq. (2.14) is substituted into Eq. (2.3), an algebraic equations' system, which can be an algebraic-differential system, is resulted. When the constructed system's numerator is forced to be vanished and substituted our results into Eq. (2.14), the exact solution is determined for the studied problem.

3 The Ito Equations' Applications

The (2+1)-D non-local Ito equation is investigated via the Kudryashov SEM, which was discussed in the previous section. With the application of the transformation given in Eq. (2.2), Eq. (1.2) will be carried into the following ODE form:

$$(\eta - \alpha - \beta)\, v^{(3)} - v^{(5)} - 6\left((v'')^2 + v'v^{(3)}\right) = 0. \tag{3.1}$$

In further compact form, Eq. (3.1) can be expressed as:

$$(\eta - \alpha - \beta)\, v^{(3)} - v^{(5)} - 3\left((v')^2\right)'' = 0. \tag{3.2}$$

By the twice integrartion Eq. (3.2) w.r.t ξ and let the integration's constants to be 0, we have:

$$(\eta - \alpha - \beta)\, v' - v^{(3)} - 3(v')^2 = 0. \tag{3.3}$$

Let $w(\xi) = v'(\xi)$ to get:

$$(\eta - \alpha - \beta)\, w - w'' - 3w^2 = 0. \tag{3.4}$$

The balance is made between w'' and w^2 in Eq. (3.4) which $m = 2$ is obtained.

3.1 Application of SEM

As a result, Eq. (3.5) has a solution as:

$$w(\xi) = A_0 + A_1\,\phi(\xi) + A_2\,\phi(\xi)^2. \tag{3.5}$$

By the substitution of Eq. (3.5) into Eq. (3.4), and from the Bernoulli Eq. (2.5), as well as, let us set up the coefficients of ϕ^i, $i = 0, 1, \ldots, 4$, to be zero, the following system is constructed in terms of $A_0, A_1, A_2, \lambda, \mu$ and η:

$$A_0\,(3A_0 + \alpha + \beta - \eta) = 0, \tag{3.6}$$

$$A_1\left(6A_0 + \alpha + \beta - \eta + \lambda^2\right) = 0, \tag{3.7}$$

$$3A_1\,(A_1 + \lambda\mu) - A_2\left(6A_0 + \alpha + \beta - \eta + 4\lambda^2\right) = 0, \tag{3.8}$$

$$2\left(3A_1 A_2 + 5\lambda\mu A_2 + \mu^2 A_1\right) = 0, \tag{3.9}$$

$$3A_2\left(A_2 + 2\mu^2\right) = 0. \tag{3.10}$$

Solving Eq. (3.6) and Eq. (3.10) implies

$A_0 = 0$, $\frac{1}{3}(\eta - \alpha - \beta)$ and $A_2 = -2\mu^2$,

where μ is a nonzero constant. As a result, the exact traveling wave solutions of Ito Eq. (1.1) is constructed as:

Case 1. If $A_0 = 0$, $\eta = \alpha + \beta$, and $\lambda = 0$, we get $A_1 = 0$ and

$$u_{01}(x, y, t) = 2\ln(\xi_0 - \xi).$$ (3.11)

Case 2. If $A_0 = 0$, $\eta = \alpha + \beta + \lambda^2$, and $\lambda \neq 0$, we get $A_1 = -2\lambda\mu$ and

$$u_{02}(x, y, t) = -2\left(\lambda x - \ln\left(1 - \mu e^{\lambda\xi + \xi_0}\right)\right), \quad \text{for} \lambda > 0 \text{ and } \mu < 0.$$ (3.12)

$$u_{03}(x, y, t) = -2\left(\lambda x - \ln\left(1 + \mu e^{\lambda\xi + \xi_0}\right)\right), \quad \text{for } \lambda < 0 \text{ and } \mu > 0.$$ (3.13)

Case 3. If $A_0 = \frac{1}{3}(\eta - \alpha - \beta)$, $\eta = \alpha + \beta - \lambda^2$, $\lambda > 0$, $\mu < 0$ and $A_1 = -2\lambda\mu$, we get

$$u_{04}(x, y, t) = \frac{\lambda^2}{3}x\left(\frac{x}{2} - \frac{6}{\lambda} - \xi\right) + 2\ln\left(1 - \mu e^{\lambda\xi + \xi_0}\right), \quad \text{for } \lambda > 0 \text{ and } \mu < 0.$$ (3.14)

$$u_{05}(x, y, t) = \frac{\lambda^2}{3}x\left(\frac{x}{2} - \frac{6}{\lambda} - \xi\right) + 2\ln\left(1 + \mu e^{\lambda\xi + \xi_0}\right), \quad \text{for } \lambda < 0 \text{ and } \mu > 0.$$ (3.15)

As in the Bernoulli equation case, by using the Riccati equation provided in Eq. (2.9), we obtain:

$$3A_0^2 + A_0(\alpha + \beta - \eta) + 2\mu^2 A_2 = 0,$$ (3.16)

$$A_1(6A_0 + \alpha + \beta - \eta + 2\lambda\mu) = 0,$$ (3.17)

$$3A_1^2 + A_2(6A_0 + \alpha + \beta - \eta + 8\lambda\mu) = 0,$$ (3.18)

$$2A_1\left(3A_2 + \lambda^2\right) = 0,$$ (3.19)

$$3A_2\left(A_2 + 2\lambda^2\right) = 0.$$ (3.20)

Eliminating the trivial solution, Eq. (3.16) and Eqs. (2.2)-(3.20) imply that $A_2 = -2\lambda^2$, $A_1 = 0$, $A_0 = -\frac{1}{6}(\alpha + \beta - \eta + 8\lambda\mu)$ and the following cases:

Case 4. If $\eta = \alpha + \beta - 4\lambda\mu$, and $\lambda\mu < 0$, we get:

$$u_{06}(x, y, t) = 2\ln\left(\cosh\left(\sqrt{-\lambda\mu}\,\xi\right) - \frac{\varepsilon\ln(\xi_0)}{2}\right).$$ (3.21)

Or,

$$u_{07}(x, y, t) = 2\ln\left(\sinh\left(\sqrt{-\lambda\mu}\,\xi\right) - \frac{\varepsilon\ln(\xi_0)}{2}\right).$$ (3.22)

Case 5. If $\eta = \alpha + \beta + 4\lambda\mu$, and $\lambda\mu < 0$, we get:

$$u_{08}(x, y, t) = \mp\sqrt{-\lambda\mu}\,\varepsilon\ln(\xi_0)\,x + \frac{2}{3}\lambda\mu\,x(2\xi - x) + 2\ln\left(\cosh\left(\sqrt{-\lambda\mu}\,\xi\right) \pm \frac{\varepsilon\ln(\xi_0)}{2}\right).$$ (3.23)

Or,

$$u_{09}(x, y, t) = \mp\sqrt{-\lambda\mu}\,\varepsilon\ln(\xi_0)\,x + \frac{2}{3}\lambda\mu\,x(2\xi - x) + 2\ln\left(\sinh\left(\sqrt{-\lambda\mu}\,\xi\right) \pm \frac{\varepsilon\ln(\xi_0)}{2}\right).$$ (3.24)

Case 6. If $\eta = \alpha + \beta - 4\lambda\mu$, and $\lambda\mu > 0$, we get

$$u_{10}(x, y, t) = 2\ln\left(\cos\left(\sqrt{\lambda\mu}\,\xi\right) + \xi_0\right). \tag{3.25}$$

Or,

$$u_{11}(x, y, t) = 2\ln\left(\sin\left(\sqrt{\lambda\mu}\,\xi\right) + \xi_0\right). \tag{3.26}$$

Case 7. If $\eta = \alpha + \beta + 4\lambda\mu$, and $\lambda\mu > 0$, we get:

$$u_{12}(x, y, t) = 2\sqrt{\lambda\mu}\,\xi_0 x + \frac{2}{3}\lambda\mu x(2\xi - x) + 2\ln\left(\cos\left(\sqrt{\lambda\mu}\,\xi\right) + \xi_0\right). \tag{3.27}$$

Or,

$$u_{13}(x, y, t) = 2\sqrt{\lambda\mu}\,\xi_0 x + \frac{2}{3}\lambda\mu x(2\xi - x) + 2\ln\left(\sin\left(\sqrt{\lambda\mu}\,\xi\right) + \xi_0\right). \tag{3.28}$$

3.2 Application of MSEM

The MSEM is applied for Eq. (1.2) with $m = 2$, Eq. (3.4) constructs a solution in the form:

$$w(\xi) = A_0 + A_1\frac{\phi'(\xi)}{\phi(\xi)} + A_2\left(\frac{\phi'(\xi)}{\phi(\xi)}\right)^2, A_2 \neq 0. \tag{3.29}$$

It is simple to find that

$$w'(\xi) = A_1\left(\frac{\phi''}{\phi} - \left(\frac{\phi'}{\phi}\right)^2\right) + 2A_2\left(\frac{\phi'\phi''}{\phi^2} - \left(\frac{\phi'}{\phi}\right)^3\right), \tag{3.30}$$

$$w''(\xi) = A_1\left(\frac{\phi'''}{\phi} - 3\frac{\phi'\phi''}{\phi^2} + 2\left(\frac{\phi'}{\phi}\right)^3\right) + 2A_2\left(2\frac{\phi''^2 + \phi'\phi'''}{\phi^2} - 5\frac{\phi'^2\phi''}{\phi^3} + 3\left(\frac{\phi'}{\phi}\right)^4\right). \tag{3.31}$$

Substituting Eqs. (3.30)-(3.31) into Eq. (3.4) and the coefficients of ϕ^{-i} ($i = 0, \ldots, 4$) are equated to be vanished which imply the following:

$$A_0(3A_0 + \alpha + \beta - \eta) = 0, \tag{3.32}$$

$$3A_2(2 + A_2)\phi'(\xi)^4 = 0, \tag{3.33}$$

$$2\left(A_1\phi'(\xi)(1 + 3A_2) - 5\phi''(\xi)\right)\phi'(\xi)^2 = 0, \tag{3.34}$$

$$A_1\left(\phi'(\xi)(6A_0 + \alpha + \beta - \eta) + \phi'''(\xi)\right) = 0, \tag{3.35}$$

$$2A_2\phi''^2 - \left(3A_1\phi'' - 2A_2\phi'''\right)\phi' + \left(3A_1^2 + A_2(6A_0 + \alpha + \beta - \eta)\right)\phi'^2 = 0. \tag{3.36}$$

Solving Eq.(3.33), with $A_2 \neq 0$ and $\phi'(\xi) \neq 0$ to avoid trivial solution, gives that $A_2 = -2$. Accordingly, Eq.(1.1) exact travelling wave solutions are constructed as:
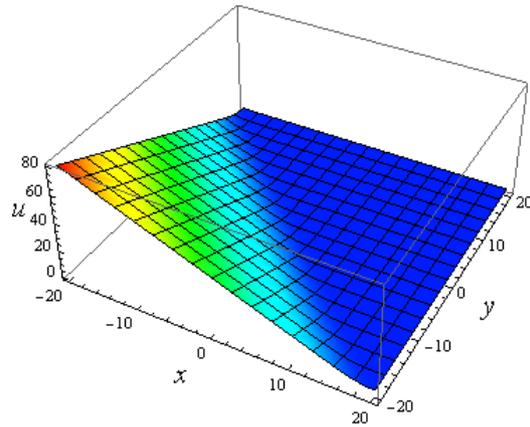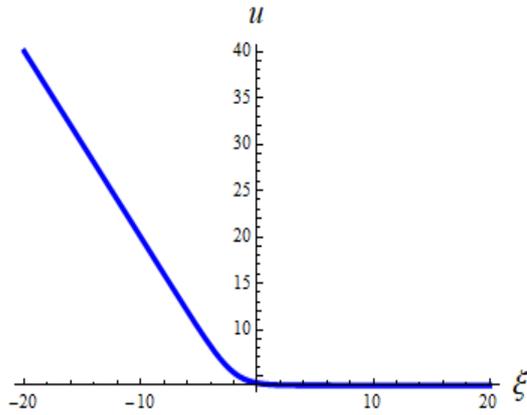
Case 1. If $A_0 = 0$, $\phi(\xi) = \frac{2e^{\frac{A_1\xi}{2}}}{A_1}C_1 + C_2$, and $\eta = \frac{1}{4}\left(A_1^2 + 4\alpha + 4\beta\right)$, we get

$$u_{14}(x, y, t) = -A_1 x + 2\ln\left(2C_1 e^{\frac{1}{2}A_1\xi} + C_2 A_1\right). \tag{3.37}$$
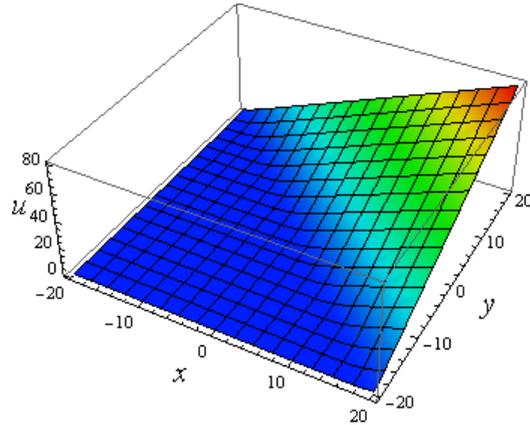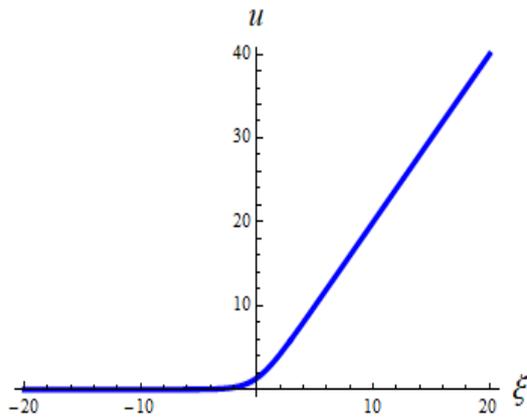
Case 2. If $A_0 = \frac{1}{3}(\eta - \alpha - \beta)$, $\phi(\xi) = \frac{2 e^{\frac{A_1 \xi}{2}}}{A_1} C_1 + C_2$, and $\eta = \frac{1}{4}\left(-A_1^2 + 4\alpha + 4\beta\right)$, we get

$$u_{15}(x, y, t) = A_1 x \left(\frac{A_1}{12}\left(\frac{x}{2} - \xi\right) - 1\right) + 2 \ln\left(2 C_1 e^{\frac{1}{2}A_1 \xi} + C_2 A_1\right), \tag{3.38}$$

where C_1 and C_2 are arbitrary constants. A_1 is nonzero arbitrary constant.

Figure 1: (a) 2D plot for u_2. (b) 3D plot for $u_2(x, y)$, when $t = 0$.

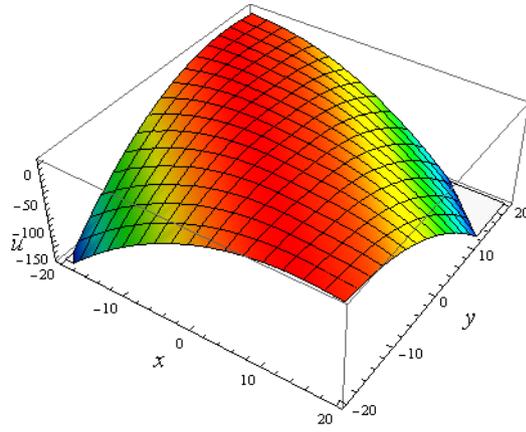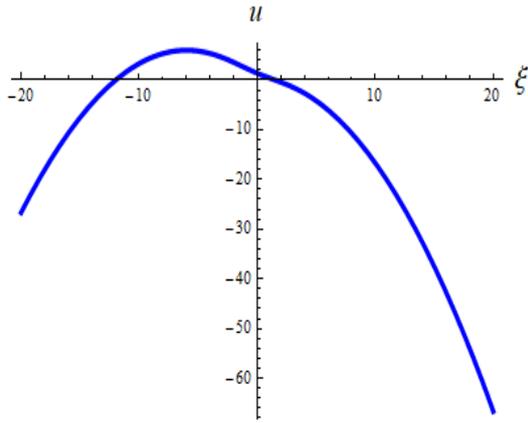**Figure 2:** (a) 2D plot for u_3. (b) 3D plot for $u_3(x, y)$, when $t = 0$.

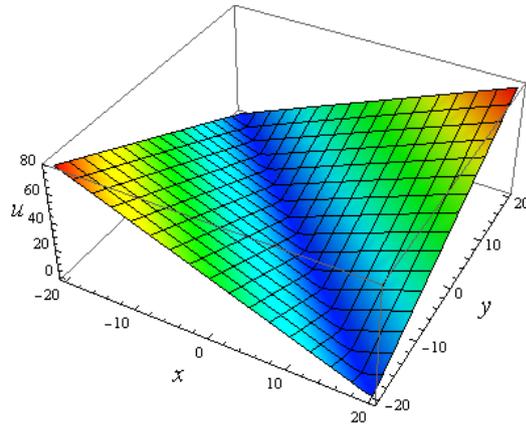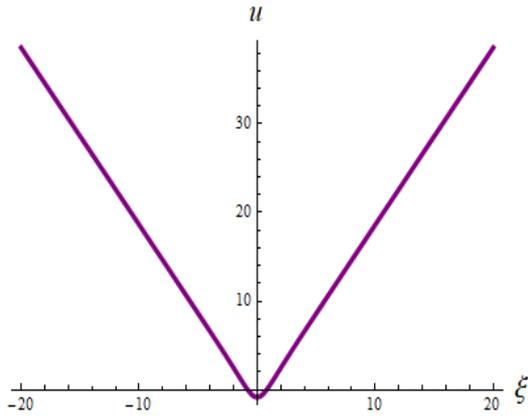**Figure 3:** (a) 2D plot for u_4. (b) 3D plot for $u_4(x, y)$, when $t = 0$.

Figure 4: (a) 2D plot for u_6. (b) 3D plot for $u_6(x, y)$, when $t = 0$.
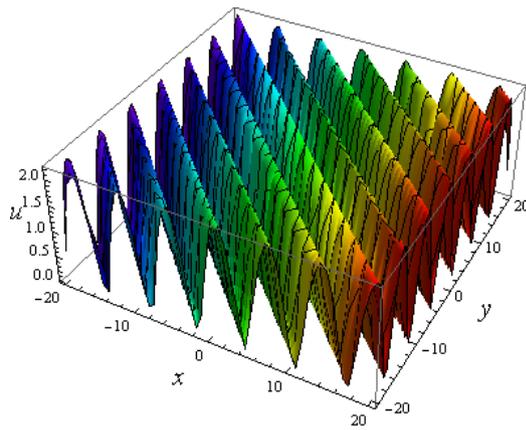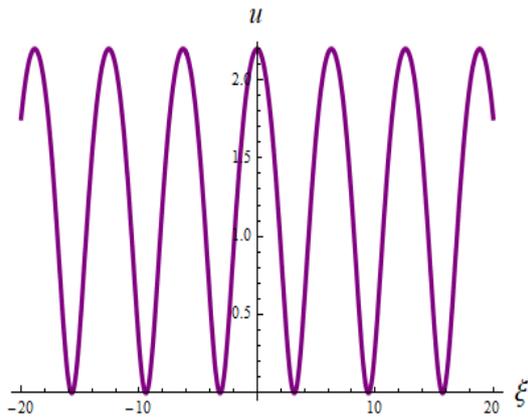

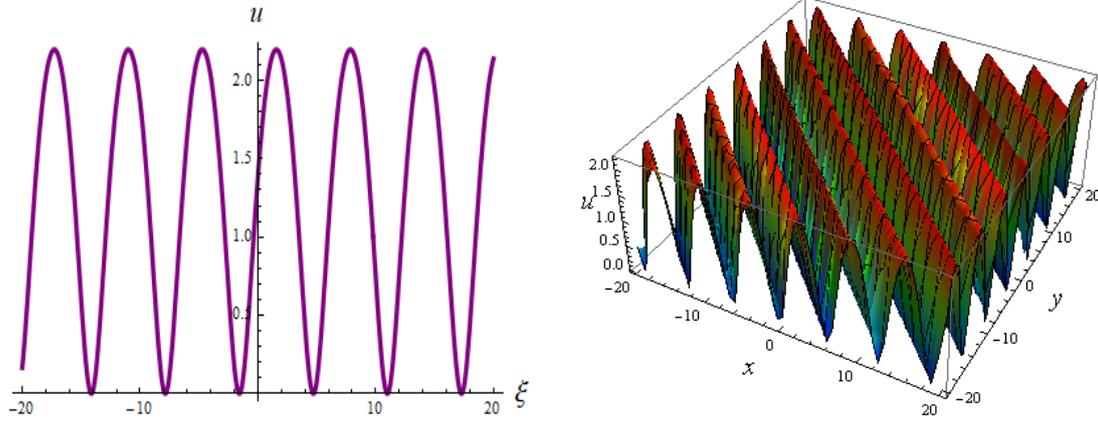**Figure 5:** (a) 2D plot for u_{10}. (b) 3D plot for $u_{10}(x, y)$, when $t = 0$.

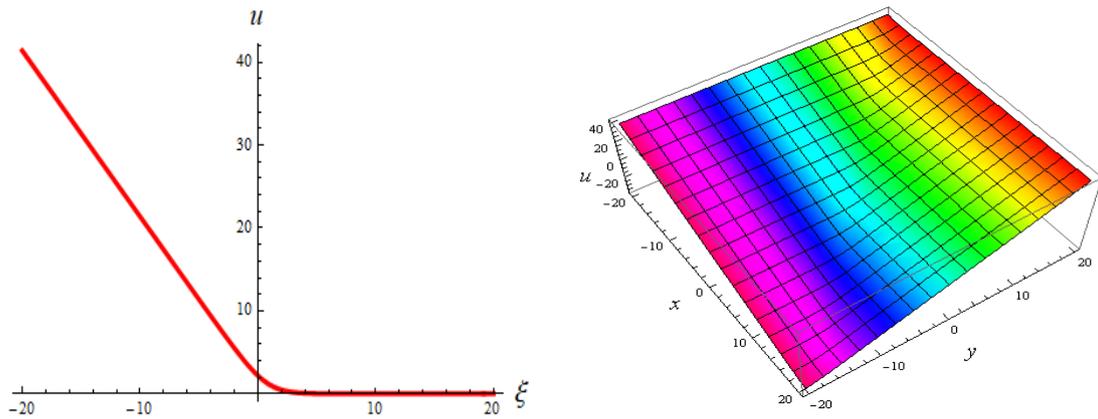**Figure 6:** (a) 2D plot for u_{11}. (b) 3D plot for $u_{11}(x, y)$, when $t = 0$.

Figure 7: (a) 2D plot for u_{10}. (b) 3D plot for $u_{10}(x, y)$, when $t = 0$.

4 Discussion and Conclusion

Three classes of traveling-wave solutions for the nonlocal (2+1) Ito equation have been explored via the SEM and MSEM. Using SEM along using of Bernoulli equation produced bright solitons provided in Eqs. (3.11)-(3.15). Similarly, with the Riccati equation, bright solitons are obtained in Eqs. (3.21)-(3.24) which include- these ones in Bernoulli equation case, singular periodic solitons in Eq. (3.25) and Eq. (3.26), and singular bright solitons in Eqs. (3.27)-(3.28) are obtained. The output of MSEM in the form of bright soliton shapes coincides with SEM associated with the Bernoulli equation via a special choice of parameters. In Figure 1, the 3-D profiles of solutions in Eq. (3.14) are shown. The singular periodic solitons given in Eq. (3.25) are plotted in Figure 2. Also, some of the solutions obtained are shown in Figures 1-7. The constructed solutions are new and have not been obtained before in any research work. The wave solutions are tested by substituting them into the fundamental equations. According to the free parameters' choice in the constructed solutions, various physical structures can result. Since solving NODEs is complicated, particularly the ones that are appeared from the MSEM, Eqs. (3.35)-(3.36), the SEM is more effective in processing the considered Ito equation reported in Eq. (1.1). The proposed techniques can be applied to various related NPDEs via Mathematica symbolic computation package 11. The considered can be investigated by using different definitions of fractional derivatives. Also, solitonic, super nonlinear, periodic, quasi-periodic, and chaotic waves can be done in future works.

References

[1] Wazwaz AM. Partial differential equations and solitary waves theory. Higher Education Press: Beijing. Berlin: Springer and Verlag; 2009. https://doi.org/10.1007/978-3-642-00251-9.

[2] Ali MN, Husnine SM. Turgut Ak, Nonlinear self- adjointness, conservation laws and exact solution of elec- trical transmission lines model. SeMA J. 2019;76:403–12.

[3] Song M, Yang CX. Exact traveling wave solutions of the Zakharov-Kuznetsov-Benjamin-Bona-Mahony equation. Appl Math Comput. 2010;216(11):3234–43.

[4] Hirota R. The direct method in soliton theory. Cambridge Tracts in Mathematics, 155. Cambridge: Cambridge University Press. 2004. https://doi.org/10.1017/CBO9780511543043.

[5] Ali MN, Seadawy AR, Husnine SM. Lie point sym- metries, exact solutions and conservation laws of Per- turbed Zakharov-Kuznetsov equation with higher order dispersion. Mod Phys Lett A. 2019;34(3):1950027.

[6] Az-Zo'bi EA. Modified Laplace decomposition method. World Appl Sci J. 2012;18(11):1481–6.

[7] Az-Zo'bi EA. New applications of Adomian decomposition method. Middle East J Sci Res. 2015;23(4):735–40.

[8] Az-Zo'bi EA. An approximate analytic solution for isen- tropic flow by an inviscid gas equations. Arch Mech. 2014;66(3):203–12.

[9] Az-Zo'bi EA. Construction of solutions for mixed hyperbolic elliptic Riemann initial value system of conservation laws. Appl Math Model. 2013;37(8):6018–24.

[10] Ozis T, Aslan I. Exact and explicit solutions to the (3 + 1)- dimensional JimboMiwa equation via the Exp-function method. Phys Lett A. 2018;372(47):7011–5.

[11] Wang M, Li X. Applications of F-expansion to periodic wave solutions for a new Hamiltonian amplitude equa- tion. Chaos Solitons Fractals. 2005;24(5):1257-1268. https://doi.org/10.1016/j.chaos.2004.09.044.

[12] Az-Zo'bi EA. On the convergence of variational iteration method for solving systems of conservation laws. Trends Appl Sci Res. 2015;10(3):157–65.

[13] Biondini G, Fagerstrom E, Prinari B. Inverse scattering trans- form for the defocusing nonlinear Schrödinger equation with fully asymmetric non-zero boundary conditions. Physica D. 2016;333:117–36.

[14] Az-Zo'bi EA, Al-Dawoud K. Semi-analytic solutions to Riemann problem for one-dimensional gas dynamics. Sci Res Essays. 2014;9(20):880–4.

[15] Az-Zo'bi EA. Analytic-numeric simulation of shock wave equa- tion using reduced differential transform method. Sci Int. 2015;273:1749–53.

[16] Az-Zo'bi EA, Al-Dawoud K, Marashdeh M. Numeric-analytic solutions of mixed-type systems of balance laws. Appl Math Comput. 2015;265:133–43.

[17] Az-Zo'bi EA. On the reduced differential transform method and its application to the generalized Burgers-Huxley equation. Appl Math Sci (Ruse). 2014;8(177):8823–31.

[18] Gao W, Rezazadeh H, Pinar Z, Baskonus HM, Sarwar S, Yel G. Novel explicit solutions for the nonlinear Zoomeron equa- tion by using newly extended direct algebraic technique. Opt Quantum Electron. 2020;52(1):1–13.

[19] Rezazadeh H, Korkmaz A, Eslami M, Mirhosseini-Alizamini SM. A large family of optical solutions to Kundu–Eckhaus model by a new auxiliary equation method. Opt Quantum Electron. 2019;51(3):84.

[20] Eslami M, Rezazadeh H. The first integral method for Wu- Zhang system with conformable time-fractional derivative. Calcolo. 2016;53(3):475–85.

[21] Az-Zo'bi EA. Analytic simulation for 1D Euler-like model in fluid dynamics. J Adv Phys. 2018;7(3):330–5.

[22] Az-Zo'bi EA. Exact series solutions of one-dimensional finite amplitude sound waves. Sci Int. 2018;30(6):817–20.

[23] Az-Zo'bi EA. Analytic treatment for generalized (m+1)-dimensional partial differential equations. J Korean Soc Ind Appl Math. 2018; 22(4);289-294. https://doi.org/10.12941/JKSIAM.2018.22.289.

[24] Az-Zo'bi EA. A reliable analytic study for higher-dimensional telegraph equation. J Math Comput Sci. 2018;18(4):423–9.

[25] Az-Zo'bi EA, Yıldırım A. The residual power series method for the one-dimensional unsteady flow of a van der Waals gas. Physica A. 2019;517:188–96.

[26] Kudryashov NA.Simplest equation method to look for exact solutions of nonlinear differential equations. Chaos Solitons Fractals. 2005; 24(2.3):1217-1231. https://doi.org/10.1016/j.chaos.2004.09.109.

[27] Kudryashov NA. Exact solitary waves of the Fisher equation. Phys Lett A. 2005;342(1-2):99–106.

[28] Biswas A, Al-Amr MO, Rezazadeh H, Mirzazadeh M, Eslami M, Zhou Q, et al. Resonant optical solitons with dual-power law nonlinearity and frac-tional temporal evolution. Optik. 2018;165:233–9.

[29] Az-Zo'bi EA. Solitary and periodic exact solutions of the vis- cosity capillarity van der Waals gas equations. Appl Appl Math. 2019;14(1):349–58.

[30] Al-Amr MO. Exact solutions of the generalized (2+1)- dimen- sional nonlinear evolution equations via the modified simple equation method. Comp Math Appl. 2015;69(5):390–397.

[31] Irshad A, Mohyud-Din ST, Ahmed N, Khan U. A new modifica- tion in simple equation method and its applications on nonlin- ear equations of physical nature. Results Phys. 2017;7:4232–40.

[32] Arnous AH, Ullah MZ, Asma M, Moshokoa SP, Zhou Q, Mirza- zadeh M, et al. Dark and singular dispersive optical solitons of Schrödinger-Hirota equation by modified simple equation method. Optik. 2017;136:445–50.

[33] Al-Amr MO, El-Ganaini S. New exact traveling wave solutions of the (4+1)-dimensional Fokas equation. Comput Math Appl. 2017;74(6):1274–87.

[34] Vitanov NK, Dimitrova ZI, Vitanov KN. Modified method of simplest equation for obtaining exact analytical solutions of nonlinear partial differential equations: further de- velopment of the methodology with applications. Appl Math Comput. 2015;269:363–78.

[35] Vitanov NK, Dimitrova ZI. Modified method of simplest equa- tion applied to the nonlinear Schrodinger equation. J Theor Appl Mech. 2018;48(1):59–68.

[36] Biswas A, Rezazadeh H, Mirzazadeh M, Eslami M, Ekici M, Zhou Q, et al. Optical soliton perturbation with Fokas Lenells equation using three exotic and efficient integration schemes. Optik (Stuttg). 2018;165:288–94.

[37] Zayed EME, Al-Nowehy AG, Elshater MEM. Solitons and other solutions for coupled nonlinear Schrödinger equations using three different techniques. Pramana- J Phys. 2019; 92(96). https://doi.org/10.1007/s12043-019-1762-y.

[38] Vitanov NK. Modified method of simplest equation for obtain- ing exact solutions of nonlinear partial differential equations: history, recent developments of the methodol- ogy and stud- ied classes of equations. J Theor Appl Mech. 2019;49(2):107–22.

[39] Az-Zo'bi EA. Peakonand solitary wave solutions for the mod- ified Fornberg-Whitham equation using simplest equation method. Int J Math Comput Sci. 2019;14(3):635–45.

[40] Az-Zo'bi EA. New kink solutions for the van der Waals psystem. Math Methods Appl Sci. 2019;42(18):1–11.

[41] Ito M. An extension of nonlinear evolution equations of the K-dv (mK-dv) type to higher orders. J Phys Soc Jpn. 1980;49(2):771–8.

[42] Wazwaz AM. Multiple-soliton solutions for the generalized (1 + 1)-dimensional and the generalized (2 + 1)-dimensional Ito equations. Appl Math Comput. 2008;202(2):840–9.

[43] Akgul EK, Akgul A, Yavuz M. New Illustrative Applications of Integral Transforms to Financial Models with Different Frac- tional Derivatives. Chaos Solitons Fractals. 2021;146:110877.

[44] Yavuz M, Sulaiman TA, Yusuf A, Abdeljawad T. The Schrodinger-KdV equation of fractional order with Mittag- Leffler nonsingular kernel. Alex Eng J. 2021;60(2):2715–24.

[45] Yavuz M, Yokus A. Analytical and numerical approaches to nerve impulse model of fractional-order. Numer Methods Par- tial Differ Equ. 2021;36(6):1348–68.

[46] Yokus A, Yavuz M. Novel comparison of numerical and analytical methods for fractional Burger–Fisher equa- tion. Discrete Contin Dyn Syst-S. 2021;14(7):2591-2606. https://doi.org/10.3934/dcdss.2020258.

[47] Zad L, Nawaz R, Nisar KS, Tahir M, Yavuz M, Kaabar MK, et al. New approximate-analytical solutions to partial differential equations via auxiliary function method. Partial Differ Equ Appl Math. 2021;4:100045.

[48] Yavuz M, Sene N. Approximate solutions of the model de- scribing fluid flow using generalized-Laplace trans- form method and heat balance integral method. Axioms. 2020;9(4):123.

[49] Jena RM, Chakraverty S, Yavuz M. Two-hybrid techniques coupled with an integral transform for Caputo time- fractional Navier-Stokes Equations. Progr Fract Differ Appl. 2020;6(3):201–13.

[50] Abreu-Blaya R, Fleitas A, Valdés JE, Reyes R, Rodríguez JM, Sigarreta JM. On the conformable fractional logistic models. Math Methods Appl Sci. 2020;43(7):4156–67.

[51] Guzman PM, Langton G, Bittencurt LM, Medina J, Valdes Juan EN. A new definition of a fractional deriva- tive of lo- cal type. J Math Anal. 2018;9(2):88–98. Available from: www.ilirias.com/jma

[52] Fleitas A, Mendez-Bermúdez JA, Valdes JE, Almira JM. On frac- tional Liénard-type systems. Rev Mex Fis. 2019;65(6):618–25.

[53] Guzman PM, Lugo LM, Valdes JE, Vivas-Cortez M. On a new generalized integral operator and certain operating proper- ties. Axioms. 2020;9(2):69.

[54] Fleitas A, Gómez-Aguilar JF, Valdés JE, Rodríguez JM, Sigarreta JM. Analysis of the local Drude model involving the general- ized fractional derivative. Optik (Stuttg). 2019;193:163008.

[55] Baitiche Z, Derbazi C, Alzabut J, Samei ME, Kaabar MK, Siri Z. Monotone Iterative Method for -Caputo Fractional Differential Equation with Nonlinear Boundary Conditions. Fractal and Fractional. 2021;5(3):81.

[56] Alzabut J, Selvam A, Dhineshbabu R, Kaabar MK. The Exis- tence, Uniqueness, and Stability Analysis of the Discrete Frac- tional Three-Point Boundary Value Problem for the Elastic Beam Equation. Symmetry. 2021;13(5):789.

[57] Bhanotar SA, Kaabar MK. Analytical Solutions for the Nonlin- ear Partial Differential Equations Using the Conformable Triple Laplace Transform Decomposition Method. Int J Differ Equ. 2021;2021:1–18.

[58] Kaabar MK, Kaplan M, Siri Z. New Exact Soliton Solutions of the (3+1)-Dimensional Conformable Wazwaz–Benjamin– Bona–Mahony Equation via Two Novel Techniques. J Funct Spaces. 2021;2021:1–13.

[59] Martínez F. Martínez I, KaabarMKA, Paredes S. New re- sults on complex conformable integral. AIMS Math. 2020;5(6):7695–710.

[60] Martínez F, Martínez I, Kaabar MK, Paredes S. On Con- formable Laplace's Equation. Math Probl Eng. 2021;2021:1–10.

[61] Martínez F, Martínez I, Kaabar MK, Paredes S. Generalized Conformable Mean Value Theorems with Applications to Multi- variable Calculus. J Math. 2021;2021:1–7.

[62] Ma WX, Huang Y, Wang F. Inverse scattering trans- forms and soliton solutions of nonlocal reverse-space nonlinear Schrödinger hierarchies. Stud Appl Math. 2020;145(3):563–85.

[63] Ma WX. Riemann-Hilbert problems and soliton solutions of nonlocal real reverse-spacetime mKdV equations. J Math Anal Appl. 2021;498(2):124980.

[64] Ma WX. N-soliton solutions and the Hirota conditions in (1 + 1)-dimensions. Int J Nonlinear Sci Numer Simul. 2021:000010151520200214.

[65] Ma WX. N-soliton solutions and the Hirota conditions in (2+1)- dimensions. Opt Quantum Electron. 2020;52(12):511.